Asymptotic behavior of second order non-linear neutral differential equations

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Abstract

In this paper, we have given some necessary and sufficient conditions for all non-oscillatory solutions to the nonlinear neutral differential equation

\[ [y(t) - p(t)y(\tau(t))]'' + q(t)f(y(\sigma(t))) = 0 \]

So that converge to zero as \( t \to \infty \). Some examples are given to illustrate the obtained results.

1. Introduction

Consider the second order non-linear neutral differential equations

\[ [y(t) - p(t)y(\tau(t))]'' + q(t)f(y(\sigma(t))) = 0 \]  (1.1)

Under the standing hypotheses:

(A1) \( p(t) \in C([t_0, \infty); (0, \infty)), q(t) \in C([t_0, \infty); \mathbb{R}). \)

(A2) \( \tau(t), \sigma(t) \in C([t_0, \infty); \mathbb{R}^+) \), \( \lim_{t \to \infty} \tau(t) = \infty, \lim_{t \to \infty} \sigma(t) = \infty, \) and \( \tau(t), \sigma(t) \) are increasing functions.

(A3) \( f(u) \in C(\mathbb{R}; \mathbb{R}); uf(u) > 0 \) for \( u \neq 0 \); \( |f(u)| \geq \beta|u|, \beta > 0. \)

By a solution of eq.(1.1), we mean a function \( y(t) \in C([p(t), \infty); \mathbb{R}) \) such that \( y(t) - p(t)y(\tau(t)) \) is two times continuously differentiable and \( y(t) \) satisfies (1.1), where
\( \rho(t) = \min \{ \tau(t), \sigma(t), t_0 \} \). A solution \( y(t) \) is said to be oscillatory if it has arbitrarily large zeros otherwise \( y(t) \) is said to be non-oscillatory. Hence there has been much research activity concerning oscillatory and nonoscillatory behavior of solutions to different classes of neutral differential equations, we refer the reader to [1-13].

2. Main Results

Before we present the results we begin with the following lemma which is helpful to establish our main results

**Lemma 2.1** ([14], Lemma 2.1 and Lemma 2.2, pp.477-478)

Assume that \( p \in C([t_0, \infty); \mathbb{R}^+) \), \( \tau \in C([t_0, \infty); \mathbb{R}) \), for \( t \geq t_0 \),

i. Suppose that \( 0 < p(t) \leq 1 \), for \( t \geq t_0 \). Let \( y(t) \) be a non-oscillatory solution of a functional inequality \( y(t)[y(t) - p(t)y(\tau(t))] < 0 \), in a neighborhood of infinity. Suppose that \( \tau(t) < t \) for \( t \geq t_0 \), then \( y(t) \) is bounded. If moreover \( 0 < p(t) \leq \delta < 1 \), \( t \geq t_0 \), for some positive constant \( \delta \), then \( \lim_{t \to \infty} y(t) = 0 \).

ii. Suppose that \( 1 \leq p(t) \) for \( t \geq t_0 \). Let \( y(t) \) be a non-oscillatory solution of a functional inequality \( y(t)[y(t) - p(t)y(\tau(t))] > 0 \) in a neighborhood of infinity. Suppose that \( \tau(t) > t \) for \( t \geq t_0 \), then \( y(t) \) is bounded. If moreover \( 1 \leq p(t) \), \( t \geq t_0 \), for some positive constant \( \delta \), then \( \lim_{t \to \infty} y(t) = 0 \).

Let

\[ z(t) = y(t) - p(t)y(\tau(t)) \tag{1.2} \]
Theorem 2.2 Assume that \((A_1) - (A_3)\) hold, \(p(t) \geq p > 1, q(t) < 0, \sigma^{-1}(\tau(t)) > t\)
\(\tau(t) > t, \) and

\[
\limsup_{t \to \infty} t \int_{\sigma^{-1}(\tau(t))}^{\infty} \frac{|q(s)|}{p(\sigma^{-1}(\sigma(s)))} ds > \frac{1}{\beta} \tag{1.3}
\]

Where \(\beta\) as \(\text{in}(A_3)\). Then every nonoscillatory solution of equation (1.1) tends to zero as \(t \to \infty\).

**Proof.** Suppose that \(y(t)\) be an nonoscillatory solution of (1.1). Without loss of
generality assume that \(y(t) > 0, y(\sigma(t)) > 0, y(\tau(t)) > 0\) for \(t \geq t_0\).

Then from (1.1) and (1.2) it follows that

\[
z''(t) = -q(t)f(y(\sigma(t))) \geq 0 \tag{1.4}
\]

Hence \(z(t), z'(t)\) are monotone functions, we have two cases for \(z'(t)\)

1. \(z'(t) > 0\) for \(t \geq t_1 \geq t_0;\)
2. \(z'(t) < 0\) for \(t \geq t_1 \geq t_0;\)

**Case 1:** In this case \(z''(t) \geq 0, z'(t) > 0, z(t) > 0\), leads to \(\lim_{t \to \infty} z(t) = \infty\).

Then from (1.2) it follows that \(z(t) \leq y(t)\) which implies that \(\lim_{t \to \infty} y(t) = \infty\).

On the other side by lemma[2.1-ii], it follows that \(y(t)\) is bounded, this is a
contradiction.

**Case 2:** \(z''(t) \geq 0, z'(t) < 0\), we have two sub-cases for \(z(t)\)

Case (a) \(z(t) > 0\) for \(t \geq t_2 \geq t_1;\) Case (b) \(z(t) < 0\) for \(t \geq t_2 \geq t_1;\)
Case (a): In this case we have $z''(t) \geq 0$, $z'(t) < 0$, $z(t) > 0$.

By lemma [2.1-ii], it follows that $\lim_{t \to \infty} y(t) = 0$.

Case (b) $z''(t) \geq 0$, $z'(t) < 0$, $z(t) < 0$

From (1.2) we get $z(t) > -p(t)y(\tau(t))$ that is $y(\tau(t)) > \frac{-1}{p(\tau(t))} z(t)$

then

$$y(\sigma(t)) > \frac{-1}{p \left( \tau^{-1}(\sigma(t)) \right)} z(\tau^{-1}(\sigma(t)))$$

(Integrating (1.4) from $t$ to $\infty$ we get)

$$-z'(t) \geq - \int_t^\infty q(s)f \left( y(\sigma(s)) \right) ds$$

(1.6)

Using $(A_3)$ in (1.6) it follows that

$$-z'(t) \geq -\beta \int_t^\infty q(s)y(\sigma(s)) ds$$

(1.7)

Substituting (1.5) in (1.7) we obtain

$$-z'(t) \geq \beta \int_t^\infty \frac{q(s)}{p(\tau^{-1}(\sigma(s)))} z(\tau^{-1}(\sigma(s))) ds$$

(1.8)

Now from condition (1.3) we have

$$1 < \beta t \int_t^\infty \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds \leq \beta \int_{\sigma^{-1}(\tau(t))}^\infty \frac{s|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds$$

We claim that the condition (1.3) implies that

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\[ \int_{t_2}^{\infty} \frac{s|q(s)|}{p(\tau^{-1}(\sigma(s)))} \, ds = \infty, \text{ for } t \geq t_2 \]  

(1.9)

Otherwise

\[ \int_{t_2}^{\infty} \frac{s|q(s)|}{p(\tau^{-1}(\sigma(s)))} \, ds < \infty \]

We can choose \( t_3 \geq t_2 \) large enough such that

\[ \int_{t_3}^{\infty} \frac{s|q(s)|}{p(\tau^{-1}(\sigma(s)))} \, ds < 1 \] which is a contradiction.

Multiplying (1.4) by \( t \) and integrating from \( t_2 \) to \( t \), we have for all \( t \geq t_2 \),

\[ \int_{t_2}^{t} z''(s) \, ds = - \int_{t_2}^{t} q(s)f(y(\sigma(s))) \, ds \]

\[ tz'(t) - t_2z'(t_2) - z(t) + z(t_2) \geq -\beta \int_{t_2}^{t} s q(s)y(\sigma(s)) \, ds \]

\[ \geq \beta z(\tau^{-1}(\sigma(t_2))) \int_{t_2}^{t} \frac{s q(s)}{p(\tau^{-1}(\sigma(s)))} \, ds \]

As \( t \to \infty \) the last inequality yields to

\[ \lim_{t \to \infty} [tz'(t) - z(t) - t_2z'(t_2) + z(t_2)] \geq \beta z(\tau^{-1}(\sigma(t_2))) \int_{t_2}^{\infty} \frac{s q(s)}{p(\tau^{-1}(\sigma(s)))} \, ds \]

Then \( \lim_{t \to \infty} [tz'(t) - z(t)] = \infty \)

Hence

\[ tz'(t) \geq z(t) \quad \text{for} \quad t \geq t_3 \geq t_2 \]  

(1.10)

From (1.8) we get
-t \frac{q(s)}{p(\tau^{-1}(\sigma(s)))} z(\tau^{-1}(\sigma(s))) ds, \quad (1.11)

Substation (1.10) in (1.11) we get

\begin{align*}
-\frac{\beta t}{\int_{\sigma^{-1}(\tau(z))}^{\infty}} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} z(\tau^{-1}(\sigma(s))) ds \\
\geq -\beta t z(z(t)) \int_{\sigma^{-1}(\tau(z))}^{\infty} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds
\end{align*}

\begin{align*}
1 \geq \beta t \int_{\sigma^{-1}(\tau(z))}^{\infty} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds
\end{align*}

Which is a contradiction. The proof is complete. \hfill \Box

**Example 1:** Consider the following neutral delay equation

\begin{align*}
\left[ y(t) - \left( \frac{3}{2} - \frac{1}{8t} \right) y(2t) \right]'' - \frac{1}{6t^2} \left[ 1 + \frac{1}{4} y(\frac{t}{3}) \right] y(\frac{t}{3}) = 0, \quad t > \frac{1}{2} \quad (E1)
\end{align*}

\begin{align*}
p(t) = \frac{3}{2} - \frac{1}{8t}, \quad \tau(t) = 2t, \quad \sigma(t) = \frac{t}{3}, \quad q(t) = \frac{-1}{6t^2}, \quad f(y(\sigma(t))) = \left[ 1 + \frac{1}{4} y(\frac{t}{3}) \right] y(\frac{t}{3}),
\end{align*}

\begin{align*}
\sigma^{-1}(\tau(t)) = 6t, \quad \tau^{-1}(\sigma(s)) = \frac{t}{6} \text{ and } \beta = 1.
\end{align*}

\begin{align*}
\int_{\sigma^{-1}(\tau(z))}^{\infty} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds &= \int_{\frac{1}{8}}^{\infty} \frac{1}{2s^3} ds \\
\limsup_{t \to \infty} \beta t \int_{\sigma^{-1}(\tau(z))}^{\infty} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds &= \lim_{t \to \infty} \frac{2}{9} \lim_{t \to \infty} \int_{2s}^{\infty} \frac{2}{2s-1} - \frac{1}{s} ds = \infty
\end{align*}
All condition of theorem 2.2 hold. Then every solution of \((E1)\) tends to zero as \(t \to \infty\). For instance \(y(t) = \frac{1}{t}\) is such a solution.

Theorem 2.3 Assume that \(0 < p(t) \leq p, q(t) \geq 0, \tau(t) < t, \sigma^{-1}(\tau^n(t)) > t\) and

\[
\limsup_{t \to \infty} \beta t \int_{\tau^{-n}(\sigma(s))}^{\infty} f(1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^n(\sigma(s)))) q(s) \, ds > 1 \quad (1.12)
\]

Where \(\beta\) as in \((A_3)\). Then every nonoscillatory solution of \((1.1)\) tends to zero as \(t \to \infty\).

Proof. Suppose that \(y(t)\) be nonoscillatory solution of \((1.1)\). Without loss of generality assume that \(y(t)\) is an eventually positive so there exists \(t_0\) such that \(y(t) > 0, y(\tau(t)) > 0\) and \(y(\sigma(t)) > 0\) for \(t \geq t_0\).

From \((1.1)\) it follows that

\[
z''(t) = -q(t)f(y(\sigma(t))) \leq 0 \quad (1.13)
\]

Then we have two cases for \(z'(t)\)

Case 1. \(z'(t) < 0\) for \(t \geq t_1 \geq t_0\); Case 2. \(z'(t) > 0\) for \(t \geq t_1 \geq t_0\).

Case 1: We have \(z''(t) \leq 0, z'(t) < 0, z(t) < 0\), it follows that

\[
\lim_{t \to \infty} z(t) = -\infty , \text{since } z(t) > -p(t)y(\tau(t)) \text{ then } \lim_{t \to \infty} y(t) = \infty.
\]
On the other hand by lemma [2.1-i] it follows $y(t)$ is bounded, which is a contradiction.

**Case 2:** In this case $z''(t) \leq 0$, $z'(t) > 0$, we have two sub-cases for $z(t)$

Case (a) $z(t) < 0$ for $t \geq t_2 \geq t_1$; Case (b) $z(t) > 0$ for $t \geq t_2 \geq t_1$.

**Case (a):** $z''(t) \leq 0$, $z'(t) > 0$, $z(t) < 0$

By lemma[ 2.1-i] it follows that $\lim_{t \to \infty} y(t) = 0$.

**Case (b):** $z''(t) \leq 0$, $z'(t) > 0$, $z(t) > 0$

\[
y(t) = z(t) + p(t)y(\tau(t)) = z(t) + p(t)[z(\tau(t)) + p(\tau(t))y(\tau^2(t))] = z(t) + p(t)z(\tau(t)) + p(t)p(\tau(t))[z(\tau^2(t)) + p(\tau^2(t))y(\tau^3(t))]\]

As in thermo [2.4] From [15] we can written the following inequality

\[
y(\sigma(t)) \geq f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))] z\left(\tau^n(\sigma(t))\right) \tag{1.14}\]

Integrating (1.13) from $t$ to $\infty$ we get

\[-z'(t) \leq -\int_{t}^{\infty} q(s)f(y(\sigma(s)))ds\]

Using \((A_2)\) the last inequality implies

\[-z'(t) \leq -\beta \int_{t}^{\infty} q(s)y(\sigma(s))ds \tag{1.15}\]

Using (1.14) we get
\[-z'(t) \leq -\beta \int_t^\infty q(s)f[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))]z(\tau^n(\sigma(s)))ds\]  
(1.16)

We claim that

\[\int_t^\infty sf[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))]q(s)ds = \infty\]  
(1.17)

Otherwise

\[\int_{t_2}^\infty sf[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))]q(s)ds < \infty \text{ for } t > t_2\]

We can find \( t_* > t_2 \) large enough such that

\[\int_{t_*}^\infty sf[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))]q(s)ds < 1\]

Which is a contradiction. Then (1.17) holds.

From (1.13) we obtain

\[\int_{t_2}^\xi s z''(s)ds = -\int_{t_2}^\xi s q(s)f(y(\sigma(s)))ds\]

\[tz'(t) - tz'(t_2) - z(t) + z(t_2) \leq -\beta \int_{t_2}^\xi s q(s)y'(\sigma(s))ds\]

\[\leq -\beta \int_{t_2}^\xi sq(s)f[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))]z(\tau^n(\sigma(s)))ds\]

\[\leq -\beta z(\tau^n(\sigma(t_2)))\int_{t_2}^\xi sf[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))]q(s)ds\]
hence as $t \to \infty$ the last inequality implies that

$$\lim_{t \to \infty} [z(t) - tz'(t)] = \infty$$

Then for $t \geq t_3 \geq t_2$

$$z(t) > tz'(t) \quad (1.18)$$

From (1.16)

$$tz'(t) \geq \beta t \int_t^\infty q(s)f[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))z(\tau^n(\sigma(s)))ds \quad (1.19)$$

Substituting (1.18) in (1.19) we get

$$z(t) \geq \beta t \int_t^\infty q(s)f[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))z(\tau^n(\sigma(s)))ds$$

$$\geq \beta t \int_{\sigma^{-1}(\tau^{-n}(t))}^\infty q(s)f[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s)))z(\tau^n(\sigma(s)))ds$$

$$\geq \beta tz(t) \int_{\sigma^{-1}(\tau^{-n}(t))}^\infty q(s)f[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s))) ds$$

$$1 \geq \beta \int_{\sigma^{-1}(\tau^{-n}(t))}^\infty q(s)f[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(s))) ds \quad \text{which is a contradiction.}$$

The proof is complete. \quad \Box

**Example 2:** Consider the following neutral delay equation

$$[y(t) - 8y(\sqrt{t})]'' + \left[\frac{8}{t^2} - \frac{3}{t^3}\right] \sqrt{y\left(\frac{t^2}{4}\right)} = 0, \quad t \geq \frac{1}{2} \quad (E2)$$
\[ \beta = 1, n = 1, \sigma(t) = \frac{t^2}{2}, \tau(t) = \sqrt{t}, \sigma^{-1}(\tau^{-1}(t)) = 2t, p(t) = 8, q(t) = \frac{8}{t^2} - \frac{3}{t^2}, \]

\[ f(y(t)) = \sqrt{y(t)}, \quad f\left[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))\right] = \sqrt{3} \]

\[ \int_{\sigma^{-1}(\tau^{-n}(t))}^{\infty} f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))] q(s) \, ds = \sqrt{3} \int_{2t}^{\infty} \left[ \frac{8}{s^2} - \frac{3}{s^3} \right] \, ds \]

\[ \limsup_{t \to \infty} \beta t \int_{\sigma^{-1}(\tau^{-n}(t))}^{\infty} f(1 + \sum_{i=0}^{n} \prod_{k=1}^{i-1} p(\tau^k(\sigma(s))) q(s) \, ds = \sqrt{3} \lim_{t \to \infty} \left[ 4 - \frac{3}{\beta t} \right] = 4\sqrt{3} > 1 \]

All condition of theorem [2.3] hold. Then every solution of (E2) tends to zero as \( t \to \infty \). For instance \( y(t) = \frac{1}{t^2} \) is such a solution.

**References**


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السلوك المحاذي للمعادلات التنافضية التباطؤية المحايدة غير الخطيه من الرتبة الثانية

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الخلاصة :
قدمنا في هذا البحث بعض الامثلة التوضيحية للحلول غير المتذبذبة للمعادلة التباطؤية غير الخطيه من الرتبة الثانية

\[ y(t) - p(t)y(\tau(t)) \]'' + q(t)f(y(\sigma(t))) = 0

التي تضمن تقارب هذه الحلول الى الصفر عندما \( t \to \infty \) وكما قدمنا بعض الامثلة التوضيحية اللتي تحقق النتائج التي حصنا عليها.