Modules and Bounded Linear Operators

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Abstract
An associated R-module of T, which is denoted by V_{T,T^*} is given, Where V is an inner product space and T is bounded linear operator on V. We study in this paper properties of T which effects V_{T,T^*} and conversely.

Keywords: The module of an operator, algebraic elements of T, torsion elements of H_{T,T^*}.

1. Introduction
SALMA M. FARIS in [1] described a left R-module V where R the polynomials ring in x and V is vector space as follows:- \( \Phi : R \times V \rightarrow V \) by \( \Phi(P, v) = P.v = P(T)v \) this function makes V a left R -module denoted by V_T. In this paper we start by introducing a left R-module on the ring of polynomials in x, y and V is an inner product space as follows:- \( \Psi : R \times V \rightarrow V \) by \( \Psi(P, v) = P(T,T^*)v \) this function makes V a left R-module and denote this module by V_{T,T^*}. In proposition (3.2) we give form of elements of V_{T,T^*}. We prove that V_{S,S^*} is isomorphic to V_{T,T^*} if and only if S is similar to T, we study the relation between the \(*-\) algebraic elements and the torsion elements of H_{T,T^*} and the module associated with the unilateral shift operator we prove that H_{U,U^*} is acyclic R-module.

2. Preliminaries
In this section the fundamental basic concepts and primitive results are Given.

Definition (2.1) [1]:
Let V be a vector space over a field F. Let T be a linear operator acting on the elements of V on the left. Let R = F[x] be the ring of polynomials in x with coefficients in F. Define \( \Phi: R \times V \rightarrow V \) by \( \Phi(P, v) = P.v = P(T)v \).

It is clear that \( \Phi \) makes V a left R-module denoted V_T, and call it the associated R-module.

The form of every element in V_T is illustrated in the following proposition.

Proposition (2.2) [1]:
If S = \{V_j : j \in \Lambda\} is a basis for V, then each element of V_T can be written in the form \( \sum_{i=0}^{n} \sum_{j \in \Lambda} c_{ij} T^i v_j \), where \( c_{ij} \in F \).

The symbol \( \sum_{j \in \Lambda} \) means that the sum is taken over a finite subset of \( \Lambda \).

Remark (2.3) [1]:
\( V_I \) is a finitely generated R-module if and only if V is a finite dimensional vector space.
In this remark there is a relation between a finite dimensional vector space V and V_T.

Remark (2.4) [1]:
Let V be a finite dimensional vector space. Let T be an operator on V, then V_T is a finitely generated R-module.
Recall that if T and S two operators on \( V,S \) is similar to T if there exists an invertible operator h on V such that \( hSh^{-1} = T \) [2].

Proposition (2.5) [1]:
Let T and S be two operators on V. Then \( V_S \) is isomorphic to \( V_T \) if and only if S is similar to T.

Definition (2.6) [2]:
Let T be an operator on a vector space. T is said to be of finite rank if the image of T is finite dimensional.
It is shown in (2.4) that if V is a finite dimensional vector space, then V_T is a finitely generated R-module. Also if V is finite dimensional vector space, and T is any operator on V, then TV is finite dimensional. Hence T is of finite rank. Following proposition give the converse.

Proposition (2.7) [1]:
If T is of finite rank, and V_T is finitely generated, then V is finite dimensional.
Definition (2.8) [3]:
Let \( T: V \rightarrow V \) be an operator. \( v \in V \) is said to be an algebraic element (or \( T \)-algebraic) if there exists a non zero polynomial \( P \in \mathbb{R} \) such that \( P(T)v = 0 \).
\( T \) is said to be algebraic if there exists \( P \neq 0 \) in \( \mathbb{R} \) such that \( P(T)v = 0, \forall v \in V \).

Proposition (2.9)[1]:
Let \( T: V \rightarrow V \) be an operator. Let \( A = A(T) \) be the set of all \( T \)-algebraic elements. Then \( A \) is a subspace of \( V \).
There is a relation between the \( T \)-algebraic elements and the torsion elements of \( V_T \) this relation is studied in the next proposition.
Recall that an element \( m \) of \( S \)-module where \( S \) is a ring is torsion element if there exists \( 0 \neq t \in S \) such that \( tm = 0 \), \( M \) is torsion \( S \)-module if \( \tau(M) = M \) where \( \tau(M) \) the set of all torsion elements .[3]

Proposition (2.10)[1]:
Let \( T \) be an operator on \( V \) then \( A_T = \tau(V_T) \).
Recall that for any ring \( S \) and any \( S \)-module \( M, \text{ann} (M) = \{ t \in S : tm = 0, \forall m \in M \} \), and \( \text{ann} (M) = 0 \) then \( M \) is a faithful \( S \)-module.[4]

Proposition (2.11)[1]:
\( V_T \) is faithful \( R \)-module if and only if \( T \) is not an algebraic operator.
The module of the Unilateral shift operator is given finally.
Let \( U: l_2(R) \rightarrow l_2(R) \) be the operator defined by \( U(x_1, x_2, ...) = U(0, x_1, x_2, ...) \)
This operator called the Unilateral shift operator.[5]

Remark (2.12)[1]:
\( \forall i, k \in N \), one can easily see that:
1. \( U e_k = e_{k+1} \).
2. \( U^i e_k = e_{i+k} \).
3. \( U^j e_k = U^{j+k-1} e_1 \).
Recall that a left \( R \)-module \( M \) is said acyclic if \( M \) can be generated by a single element. \( M(x) = Rx = \{ r x / r \in R \} \) for some \( x \in M \).

Theorem (2.13)[1]:
Let \( U \) be the Unilateral shift operator on \( H \).
Then \( H_U \) is a cyclic faithful \( R \)-module. Hence a free \( R \)-module.

3. Main Results
Definition (3.1): Let \( R = \mathbb{F}[x, y] \) be the ring of polynomials in \( x, y \) with coefficients in \( F \). Let \( V \) be an inner product space over afield \( F \) and \( T \) be a bounded linear operator acting on the elements of \( V \) on the left .We will define a left \( R \)-module on \( V \) as follows: \( \Psi: R \times V \rightarrow V \)
byp\( \Psi(P, v) = (P(T), \tau)\)v \( i.e \) \( P(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j, a_{ij} \in F \). [6] It is clear that \( \Psi \) makes \( V \) a unitary left \( R \)-module. We shall denote this module by \( V_{T,T^*} \).
In the following proposition we introduce the form of each element of \( V_{T,T^*} \).

Proposition (3.2):
If \( S = \{ v_i : i \in \Lambda \} \) is a basis for \( V \), then each element of \( V_{T,T^*} \) can be written in the form \( \sum_{j=0}^{m} \sum_{i=0}^{n} c_{it} T^{i} T^{*j} v_i, c_{it} \in F \)
The symbol \( \sum_{i \in \Lambda} \) means that the sum is taken over a finite subset of \( \Lambda \).

Proof:- let \( w \in V_{T,T^*} \), then \( w = \sum_{k=1}^{m} \sum_{j=0}^{m} b_{kl} v_i \), where \( \sum_{k=1}^{m} \sum_{j=0}^{m} a_{ik} T^i T^{*j} (\sum_{l \in \Lambda} b_{kl} v_i) \)
Let \( n = \max \{ n_1, n_2, \ldots, n_m \} \), \( a_{ik} = 0, \forall i > n_k, k = 1, 2, \ldots, m \).

Then \( w = \sum_{j=0}^{m} \sum_{i=0}^{n} a_{ij} T^i T^{*j} \sum_{l \in \Lambda} b_{kl} v_i \)
Examples (3.3):
1. Let \( \{ v_i : i \in \Lambda \} \) be a basis for an inner product space \( V \).
(a) Let \( 0 \) be the zero operator on \( V \).If \( w \in V_{0,0} \), then by proposition (3.2)
This shows that $V_{1,1^*}$ is a finitely generated $R$-module.

Compare the following with proposition (2-5)

**Proposition (3.5):**

Let $T, S$ be two bounded operators on $V$, then $V_{S,S^*}$ and $V_{T,T^*}$ are isomorphic $R$-module iff $S$ and $T$ are similar.

**Proof:**

If $V_{S,S^*}$ is isomorphic to $V_{T,T^*}$

Let $h: V_{S,S^*} \to V_{T,T^*}$ be an $R$-isomorphism

Thus $h(w_1 + w_2) = h(w_1) + h(w_2), \forall w_1, w_2 \in V_{S,S^*}$

$h(P(x,y) \cdot w) = P(x,y) \cdot h(w), \forall P \in R, w \in V_{S,S^*}$

i.e $h$ is homomorphism. then we can define $h$ as:

$h[P(S, S^*)w] = P(T, T^*)h(w)$

If $P$ is a constant polynomial $a, a \in F$, then $h(av) = ah(v)$

Thus $h$ is a linear operator call it again $h, if \n

If $S$ and $T$ are similar then there exists an operator $h$ on $V s.t$

$h(S + S^*)^{-1} = T + T^*$ it is easy to cheack that

$hP(S, S^*) \cdot h^{-1} = P(T, T^*)h \forall P \in R$ .................. (1)

Define $h': V_{S,S^*} \to V_{T,T^*}$

By $h' [P(S, S^*)v] = P(T, T^*)h(v)$ .................. (2)

If $P_1(S, S^*)v_1 = P_2(S, S^*)v_2$

Then $h[P_1(S, S^*)v_1] = h[P_2(S, S^*)v_2]$ (since $h$ operator)

Then by

(1) $P_1(T, T^*)h(v) = P_2(T, T^*)h(v_2)$

By

(2) $h'[P_1(S, S^*)v_1] = h'[P_2(S, S^*)v_2].thus \ h'$ is well define.

If $h'[P(S, S^*)v] = 0$ , then $P(T, T^*)h(v) = 0$

By (1) $hp(S, S^*)v = 0$ but $h$ is invertible then $p(S, S^*)v = 0$
Therefore h' is 1-1
Let P(T, T*)v ∈ V_{T, T*} since v ∈ V
Then h^{-1} (v) ∈ V and P(S, S*)h^{-1}(v) ∈ V_{S, S*}

h'[P(S, S*)h^{-1}(v)] = P(T, T*)hh^{-1}(v) = P(T, T*)v

Thus h' is on to
Note h'[P(S, S*)v] = h[P(S, S*)v], but h is an
operator, hence h' is an R-homomorphism, therefore h' is an
R-isomorphism.

Remark (3.6):
If V is a finite dimensional an inner product
space, then V_{T, T*} is finitely generated
R-module.
We show in (3.6) that if V is a finite
dimensional an inner product space, then V_{T}

is finitely generated R-module, also if V is finite
dimensional and T is any operator on V, then
TV is finite dimensional, hence T is of finite
rank.

Proposition (3.7):
If T is of finite rank, and V_{T, T*} is finitely
generated, then V is finite dimenional.

Proof:
Let K = K(T T*) = \{w ∈ V: TT*w = 0\} it
is clear that K is an invariant subspaces of V,
and TT* V ≅ \frac{V}{K}

We prove by contradiction way .Assume V
is not finite dimensional. TT*V is finite
dimensional since T is finite rank,thus K must
be infinite dimensional but K is an invariant
subspace of V,then the submodule K_{T, T*} is

generated by the set \{T^{i}T^{j}w_{l}: i ∈ \Lambda; i = 0, 1, \cdots; j = 0, 1, \cdots\} where \{w_{l}: l ∈ \Lambda\}
is abasis for K.w_{l} ∈ k means that T T^{*}w_{l} = 0 .Hence the restriction of TT* on K is the
zero operator,thus K_{T, T*} = K_{0, 0}^{*} by (3.2) K_{T, T*}
cannot be finitely generated, and since R

Noetherian [7] ,V_{T, T*} is finitely generated then
K_{T, T*} is finitely generated .this contradiction
shows that V is finite dimensional.

Definition (3.8) [8]:
An operator T ∈ B(H) is said to be
* -algebraic operator if there exists non-zero polynomial of two variables P such that

P(T, T^*)x = 0 , ∀x ∈ H . x ∈ H is called
*-algebraic element if there exists non zero
polynomial of two variables P such that
P(T, T^*)x = 0.

Proposition (3.9):
Let T: H → H and A = A(T, T^*) be the set
of all * -algebraic elements then A is a
subspace of H.

Proof:
Let u, v ∈ A then there exist non-zero
polynomial p,q inR such that

P(T, T^*)u = 0 and q(T, T^*) v = 0, then
P(T, T^*)q(T, T^*)(u+v)=0
Since R = F[x, y] is an integral domain [9],
hence Pq ≠ 0, therefore
u + v ∈ A if a ∈ F then P(T, T^*)au =
ap(T, T^*)u = 0 thus au ∈ A therefore A

subspace of H.

Proposition (3.10):
Let T be an operator on H, then
A_{T, T*} = τ(H_{T, T^*})

Proof:
let 0 ≠ w ∈ A_{T, T*}. then w = \sum_{i=0}^{n} P_{i} v_{i} for
some P_{i} ∈ R, v_{i} ∈ A ∀ i
There exists q_{i} ≠ 0 in R such that
q_{i}(T, T^*)v_{i} = 0
henceq(T, T^*) w = q . w = 0 where q = q_{1} q_{2} \cdots q_{n}Thus w ∈ τ(H_{T, T^*})
And let u ∈ τ(H),then there exists P ≠ 0 in R
Such that P . u = 0 therefore P(T, T^*)u = 0 , thus u ∈ A_{T, T^*}
Therefore A_{T, T*} = τ(H_{T, T^*})

In the following proposition we give the
relation between faithful R-module and
* -algebraic operator.

Proposition (3.11):
H_{T, T^*} is a faithful R –module if and only if
T is not * -algebraic operator.

Proof:
Let P ∈ R such that P(T, T^*)v = 0 ∀v ∈ H
Then P . v = 0 ∀v ∈ H . Thus P . v = 0 ∀v ∈
H_{T, T^*} hence P ∈ ann(H_{T, T^*})
Therefor P = 0 and T is not * -algebraic
operator.
Conversely, let P ∈ ann (H_{T, T^*})

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Then \( P.v = 0 \forall v \in H_{T,T^*} \), thus \( P(T,T^*)v = 0 \forall v \in H \).

If \( T \) is not \(*\)-algebraic operator, then \( P = 0 \). Therefore \( H_{T,T^*} \) is faithful.

Finally, we study the module of Unilateral shift operator in the following.

**Theorem (3.12):**

Let \( U \) be the Unilateral shift operator on \( H \). then \( H_{U,U^*} \) is a cyclic \( R \)-module hence a free \( R \)-module.

**Proof:**

Let \( w \in H_{U,U^*} \), then

\[
  w = \sum_{i=1}^{m} \sum_{j=0}^{m} \sum_{l=0}^{n} a_{ij}U^l U^j e_i
\]

Since \( U^* = B, w = \sum_{i=1}^{m} \sum_{j=0}^{m} \sum_{l=0}^{n} a_{ij}U^l B^j e_i \).

\[
  \sum_{i=1}^{m} \sum_{j=0}^{m} \sum_{l=0}^{n} a_{ij}U^l e_{i-j} \quad [1]
\]

Thus \( w = P.e_1 \),

where

\[
  P(x, y) = \sum_{i=1}^{m} \sum_{j=0}^{m} \sum_{l=0}^{n} a_{ij}x^{i+j} e_{i-j}
\]

Therefore \( H_{U,U^*} \) is cyclic \( R \)-module generated by \( e_1 \). Thus \( H_{U,U^*} \) is a free \( R \)-module. \( [10] \)

**Corollary (3.13):**

Let \( U \) be the unilateral shift operator on \( H \). then \( H_{U,U^*} \) is a faithful \( R \)-module.

**Proof:**

Let

\[
  P(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}x^i y^j \in \text{ann} (H_{U,U^*})
\]

then \( P(x, y) \). \( e_1 = 0 \).

Hence

\[
  \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}U^l B^j e_1 = 0, \quad \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}U^l (e_{i-j}) = 0 \quad [1]
\]

By \( (2.12) \) remark 2 we have \( \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}e_{i-j+1} = 0 \).

But \( e_1, e_2, ..., e_{m-n+1} \) are linearly independent hence \( a_{ij} = 0 \).

\( \forall i = 0, 1, ..., m, j = 0, 1, ..., n \) thus \( P = 0 \).

Therefore \( H_{U,U^*} \) is a faithful \( R \)-module.

**References**


