$*-$ Embded 0f $*-$ Semigroup into $*-$ Ring

انغمار $_*$ - شبه الزمرة في $_*$- الحلقة

Ahmed A. Omran
a.a_omran@yahoo.com
Department of Mathematics
College of Education for Pure Science
Babylon University, Babylon, Iraq

Abstract

The purpose of this paper is to study the embeddability of the $*-$ semigroup $(S,.,*)$ into $*-$ ring , and prove that embeddability referred to, is possible if the ring $(Z[S],*)$ resulted from the semigroup S and the integer ring $(Z,+,.)$ is proper involution.

الخلاصة:

الغرض من هذا البحث هو دراسة انغمار *- شبه الزمرة في * - الحلقة واثبات ان هذا الانغمار ممكن اذا كانت الحلقة $(Z[S],*)$ انعكاسية فعلية

1 Introduction

In this paper one of the important of ring is the class of $*-$semigroup ring with involution is studied .

In Section 2 all important definitions of $*-$semigroup, n-proper maximal, involution on a ring, proper involution on $*-$ring and $*-$semigroup ring are discussed.

In Section 3, we proved some propositions, remarks, and examples of the embeddable $*-$semigroup into $*-$ring.

At last, we proved that if $(S,.,*)$ is $*-$semigroup finite commutative, maybe we can be $*-$embedded into a ring involution, or we cannot be $*-$embedded into any ring with involution.

2 $*-$Semigroup Ring

Definition 2.1,[1]. Let $(S,.)$ be any semigroup. A mab $*: (S,.) \to (S,.)$ is called an involution of $(S,.)$ if the conditions are holds $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all a,b in S, $(S,.,*)$ is called $*-$semigroup with involution $*$.

Definition 2.2,[1]. $(S,.,*)$ is called n-proper if whenever $s_1s_1{}^* = s_1s_2{}^*, s_2s_2{}^* = s_2s_3{}^*, \dots, s_ns_n{}^* = s_ns_n{}^*$ implies $s_1 = s_2 = \cdots = s_n \ \forall\, s_1, s_2, \dots, s_n \in S$.

Definition 2.3,[1]. Let $(S,.,*)$ be a $*-$semigroup and $= \{s_1, s_2, \dots, s_n\} \subset S$. An element $s_k \in A$ is called maximal in A if the following two conditions are holds

1) $s_ks_k{}^* = s_ks_i{}^* \ \forall\, i = 1,2,\dots,n$ impliesed $s_k = s_i$.
2) $s_ks_k{}^* = s_{ij}{}^*(i \neq k \neq j)$ implies $s_k{}^*s_i = s_k{}^*s_j$.

Definition 2.4,[1]. Let $(R,+,.)$be a ring, an involution on this ring is a map $*: (R,+,.) \to (R,+,.)$ such that for all A, B, and C the following conditions are holds:

1) $(A+B) = A^* + B^*$
2) $(AB)^* = B^*A^*$
3) $(A^*)^* = A$

Definition 2.3,[2]. An involution $*$ on a ring R is called a proper involution if for every $A \in R$ such that $AA^* = 0$ implies $A = 0$. In this case $(R,*)$ is called a P^* −ring.

Example 2.6,[3]. $(C,*)$ is a P^* −ring where C is the complex field and $*$ is the conjugate operator.
Definition 2.7,[3].Let $(S,.,*)$ be a $*$−semigroup and let R be a $*-$ ring. We say that $(S,.,*)$ is $*$ −embedded in a $*$−ring R if there is injective map $f:(S,.,*) \to (R,+,*)$ such that $f(a.b) = f(a).f(b)$ and $f(a^*) = \left(f(a)\right)^* \ \forall\, a,b \in S$.

Definition 2.8,[4]. Let $(S,.,*)$ be a $*$−semigroup and let R be a $*$−ring
We define $R[S] = \{\sum_{i=1}^{N} a_i g_i : a_i \in R, g_i \in S, n \in Z^+\}$
Where $\sum_{i=1}^{N} a_i g_i$ is just a formal symbol.

Thus, $\sum_{i=1}^{N} a_i g_i = \sum_{i=1}^{N} b_i g_i \Leftrightarrow a_i = b_i \ \forall\, i = 1,2,\dots,n$.
Also, we define additive "+" and multiplication "." On $R[S]$ as follows.
$\sum_{i=1}^{N} a_i g_i + \sum_{i=1}^{N} b_i g_i = \sum_{i=1}^{N}(a_i + b_i) g_i$ and
$\sum_{i=1}^{N} a_i g_i \ .\sum_{i=1}^{N} b_i g_i = \sum_{i=1}^{N} c_i g_i$ where $c_i = \sum_{i=1}^{N} a_i b_i$.
The sum being taken over all pairs (a_i, b_i) such that $g_i \,.\, g_j = g_k$.
Define $*$ on $R[S]$ as $(\sum_{i=1}^{N} a_i g_i)^* = \sum_{i=1}^{N} {a_i}^* {g_i}^* \ \forall a_i \in R, g_i \in S$

Proposition 2.9. $(R[S],+,.,*)$ is a $*$−ring.
Proof. Clearly $(R[S],+)$is an abelian group and $(R[S],.)$ is semigroup under multiplication and
$\sum_{i=1}^{N} a_i g_i \ (\sum_{i=1}^{N} b_i g_i + \sum_{i=1}^{N} c_i g_i) =$
$(\sum_{i=1}^{N} a_i g_i \ .\sum_{i=1}^{N} b_i g_i) + (\sum_{i=1}^{N} a_i g_i \ .\sum_{i=1}^{N} b_i g_i)$
Thus, the left distributive law holds. Similarly the right distributive law holds.
Thus, $(R[S],+,.,*)$ is $*$−ring.
$(R[S],+,.,*)$ is called $*$−semigroup ring of $(R,+,.,*)$ over $(S,.,*)$

3 Embeddability into $(R,+,.,*)$

Proposition 3.1. If $(R[S],+,.,*)$ $*$−embeds $(S,.,*)$ through a $*$−embeding $f, w: S \to Z[S]$ is the inclution map and $g: Z[S] \to R$ is defined by
$g(\sum_{i=1}^{N} m_i s_i) = \sum_{i=1}^{N} m_i\, f(s_i) \ \ \forall m_i \in Z, s \in S$. Then the following diagram is commute

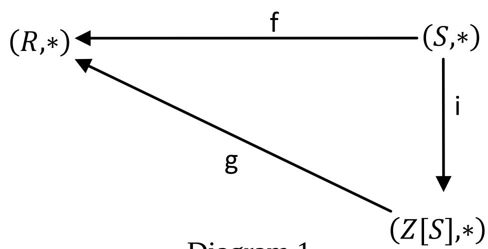

Diagram 1

Proof. Clearly g is a $*$−homomorphism and $g \circ i = f$. If $(Z[S],*)$is a proper $*$−ring , then $(S,*)$ is $*$−embeddable in P* −ring $(Z[S],*)$. It turns out that if S ia an inverse semigroup, then $(Z[S],*)$is a proper $*$−ring and $(S,*)$ is $*$−embeddable in $(Z[S],*)$.
Proposition 3.2. Let $(S,*)$ be a proper $*$−semigroup, and let $A_1 \neq A_2, A_1, A_2 \in Z[S]$ such that $A_1 {A_1}^* = A_2 {A_2}^* \in Z[S]$ and $C \in Z[S]$ such that $CC^* = m_1 A_1 + m_2 A_2$.
If $s_1 - s_2$ is a linear combination of $A_1, A_2, and\ C$ in $Z[S]$, then there is no P* −ring which $*$ $-embedding$ $(S,*)$.

Proof. Let $(R,*^{\backslash})$ be a P^* −ring, $*-embedding$ $(S,*)$ consider diagram(1) , since $g(A_1A^*{}_1) = g(A_2A^*{}_2) = 0$ in R .
$g(A_1).g(A_1)^* = g(A_2).g(A_2)^* = 0$ and hence $g(A_1) = g(A_2) = 0$.
Now $g(CC^*) = 0 = g(C).g(C)^*$ and since $(R,*)$ is P^* −ring, then $g(D) = f(s_1).f(s_2) = 0$ which implies that $f(s_1) = f(s_2)$ and so f is not injective. Thus, $(S,*)$ in not $*-embeddable$.

Proposition 3.3. Let $(S,.,*)$ be a $*-$ semigroup with n-proper maximal involution. Then $*$ is an $n-$proper involution.
Proof. Suppose that $(S,.,*)$ is $n-$proper involution, then
$s_1{s_1}^* = s_1{s_2}^*, s_2{s_2}^* = s_2{s_3}^*, \dots, s_n{s_n}^* = s_n{s_n}^*$ $\forall\, s_1, s_2, \dots, s_n \in S$. Since $(S,.,*)$ has a maximal, then there exist a maximal element $s_1 \in \{s_1, s_2, \dots, s_n\}$. From the fact
$s_i{s_i}^* = s_i{s_{i+1}}^*(mod\ n)$ implies $s_i = s_{i+1}(mod\ n)$, it follows that $s_1 = s_2 = \cdots = s_n$.

Proposition 3.4.[2]. Let $(R,*)$ be a ring with a 1-formally complex involution, then the involution is $*$ 2 –proper. Moreover if the involution $*$ is $n-$formally complex, then it is n-proper.

Corollary 3.5. Let $(R,*)$ be a ring with a formally complex involution, then the involution $*$ is $n-$proper.
Proof. Follows directly from proposition 3.4.
Remark 3.6. If $(S,.,*)$ is a finite commutative, then
1) We can be $*-embedded$ into a ring with involution, as shown in Example 3.7.
2) We cannot be $*-$ embedded into any ring with involution, as shown Example 3.8.

Example 3.7. Let $S \subset Z^2$be such that $S = \{s_1 = (1,1), s_2 = (0,1), s_3 = (-1,-1), s_1 = (0,-1)\}$. It is clear that S is semigroup under pointwise multiplication
$(a,b).(c,d) = (ab,cd)$
Define a map $*: (S,.) \to (S,.)$ by $(a,b)^* = (ab,b)$. This satisfies all conditions of involution.
It's clear that $*$ is proper (since if $s_1{s_1}^* = s_1{s_2}^* = s_2{s_2}^*, then\ s_1 = s_2$) and S is commutative.
Let $(Z[S],*)$ be $*-$semigroup ring of S over the integer where $*$ is the involution induced from S in $Z[S]$.
$X = \sum_{i=1}^{4} x_i\, s_i$ such that $XX^* = 0$, let
$f_1 = x_1^2 + x_3^2 = 0, f_2 = 2x_1x_2 + x_2^2 + 2x_3x_4 + x_4^2 = 0, f_3 = 2x_1x_3 = 0,$
$f_4 = 2x_1x_4 + 2x_3x_2 + 2x_2x_4 = 0$. It is clear that in Z if $x_1^2 + x_2^2 = 0$ then $x_1 = x_2 = 0$.
By substituting in f_2 we get $x_3 = x_4 = 0$, thus the solution of XX^* in $(Z[S],*)$ is trivial in this case. Then $(S,*)$ is $*-$ embeddable.

Example 3.8. Let $S \subset Z^3$be such that $S = \{s_1 = (-1,1,1), s_2 = (-1,1,1), s_3 = (-1,-1,1), s_1 = (1,-1,1)\}$.
Clearly S is semigroup under pointwise multiplication $(a_1,b_1,c_1).(a_2,b_2,c_2) = (a_1a_2, b_1b_2, c_1c_2)$.
Define a map $*: (S,.) \to (S,.)$ by
$(a,b,c)^* = (a,ab,c)$, this map satisfies all conditions of involution and it is proper and commutative. $X = \sum_{i=1}^{4} x_i\, s_i$ such that $XX^* = 0$, this implies
$f_1 = x_1^2 + 2x_2x_3 + x_4^2 = 0, f_2 = x_1x_3 + x_1x_2 + x_3x_4 + x_2x_4 = 0,$
$f_3 = x_2^2 + 2x_1x_4 + x_3^2 = 0,$
$f_4 = x_1^2 + 2x_1x_4 + x_3^2 = 0$
$f_1 + f_3 = (x_1 + x_4)^2 + (x_2 + x_3)^2$, $f_2 = (x_1 + x_4)(x_2 + x_3) = 0$, thus $x_1 = -x_4$, $x_2 = -x_3$, $x_2 = \pm x_1$

$X = t_1s_1 + t_1s_2 - t_1s_3 - t_1s_4$ or $X = t_1s_1 - t_1s_2 + t_1s_3 - t_1s_4 \in Z$.
Thus, the solution of the equation $XX^* = 0$ is non-trivial in this case
$A = s_1 + s_2 - s_3 - s_4$, let $B = s_1 - s_2 + s_3 - s_4$ and $C = 2s_1 + 3s_2 - 3s_3 - 2s_4$
$AA^* = 0$, $BB^* = 0$, $CC^* = -5A - 5B$
$g(AA^*) = g(BB^*) = 0$. Since g is $*-$homomorphism then $g(A) = g(B) = O_R$(since R a proper 8-ring). Hence $g(C)g(C)^* = g(CC^*) = g(-5A - 5B)$
$g(CC^*) = O_R$, $g(C) = O_R$. Let $D = C - 2A$, then $D = s_2 - s_3$
By proposition 3.1. $(S, . ,*)$ is not $* -$ embeddable.

Refrences

[1] J.Howe,"Interoduction to Semigroup Theory" Academic press(1976).
[2] M.P.Drazin,"Naturel Structures on Ring and Semigroup with Involution" Bull.Amer, Math.Soc.84(1978), 139-141.
[3] Adil A.Shehadah " n-Proper Maximal Involution on Semigroup and Ring" Math Japonica 46, NO.2(1997), 293-296.
[4] Hassan.J.S " Solution for Equtions Corresponding to $*-$ring"M.Sc at Yarmouk University,Jordan.