NUMERICAL SOLUTION FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS OF TWO-SIDED

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Abstract
In this paper we introduce the numerical solutions of two-sided fractional partial differential equations with or without parameter. The algorithm for the numerical solution for these equations is based on finite difference method. Finally, some numerical examples are provided to show that the numerical method for solving these cases is an effective solution method.

Introduction
The idea has emerged that the fractional partial differential equations, obtained from the standard partial differential equations that replace the standard partial derivatives by fractional partial derivatives, may more accurately describe some physical problems than the corresponding standard partial differential equations [1].

More and more works by researchers from various fields of science and engineering deal with dynamical systems described by fractional partial differential equations, which have been used to represent many natural processes in physics, finance, and hydrology, [2, 3, 4, 9].

Liu F. et al. [5] considered the fractional Fokker-Planck equation and presented its numerical solution. Recently, Liu F. et al. [6] also treated the fractional advection-dispersion equations and derived the complete solution of this equation with an initial condition.

In this paper presents a practical numerical method for solving the two-sided fractional partial differential equation with or without parameter of the form:

\[ \begin{align*}
\frac{\partial u(x,t)}{\partial t} & = \lambda \left[ c_+(x,t) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + c_-(x,t) \frac{\partial^\alpha u(x,t)}{\partial (-x)^\alpha} \right] \\
\frac{\partial u(x,t)}{\partial t} & = c_+(x,t) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + c_-(x,t) \frac{\partial^\alpha u(x,t)}{\partial (-x)^\alpha} + s(x,t)
\end{align*} \]

and respectively.

We use a variation on the classical explicit Euler method. We prove this method by using a novel shifted version of the usual grunwaled finite difference an approximation for the non-local fractional derivative operator.

Finite Difference Method for Solving the Two-Sided Fractional partial Differential Equations without Parameter
In this section, we use the finite difference method to solve the two-sided fractional partial differential equations of the form:

\[ \frac{\partial u(x,t)}{\partial t} = c_+(x,t) \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + c_-(x,t) \frac{\partial^\alpha u(x,t)}{\partial (-t)^\alpha} + s(x,t) \]  

subject to the initial condition
\[ u(x,0) = f(x), \quad L < x < R \]  

and the boundary conditions
\[ u(L,t) = k_1(t), \quad 0 \leq t \leq T \]  
\[ u(R,t) = k_2(t), \quad 0 \leq t \leq T \]  

where \( c_+, c. \) and \( s \) are known functions of \( x \) and \( t \), \( f \) is a known function of \( x \), \( g \) and \( k \) are known function of \( t \) and \( \alpha \) is a given fractional number.

When \( \alpha=2 \) and \( c(x,t) = c_+(x,t) + c_-(x,t) \), eq.(1) becomes the following classical parabolic partial differential equation:

\[ \frac{\partial u(x,t)}{\partial t} = c(x,t) \frac{\partial^2 u(x,t)}{\partial x^2} + s(x,t) \]

Similarly, when \( \alpha=1 \) and \( c(x,t) = c_+(x,t) + c_-(x,t) \), eq.(1) reduces to the following classical hyperbolic partial differential equation

\[ \frac{\partial u(x,t)}{\partial t} = c(x,t) \frac{\partial u(x,t)}{\partial x} + s(x,t) \]

The finite difference method starts by dividing the \( x \)-interval \([L,R]\) into \( n \) subintervals to get the grid points \( x_i = L + iAx \), where \( Ax = (R - L)/n \) and \( i = 0,1,...,n \). Also, the \( t \)-interval \([0,T]\) is divided into \( m \) subintervals to get the grid points \( t_j = j\Delta t \), \( j = 0,1,...,m \), where \( \Delta t = T/m \).

Next, by evaluating eq.(1) at \( (x_i,t_j) \) and use the explicit Euler method one can get:
Then use the shifted Grunwald estimate to the \(\alpha\)-th fractional derivative, [7]:

\[
\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{(\Delta x)^\alpha} \sum_{i=0}^{n} g_i u(x-(k-1)\Delta x,t) + O(\Delta t) \ldots \tag{3}
\]

where \(g_{i} = (-1)^{j} \frac{\alpha(a-1)-\alpha(a+k)}{k!}\), \(k=0,1,2,\ldots\)

Therefore, eq.(2) becomes

\[
\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = \frac{c_{i,j}}{(\Delta x)^{\alpha}} \sum_{k=0}^{n+1} g_{j} u_{i+k,j} + \frac{c_{i,j}}{(\Delta t)^{\alpha}} \sum_{k=0}^{n+1} g_{j} u_{i+k,j} + \Delta S_{i,j} + s_{i,j},
\]

\(i=1,2,\ldots,n-1, j=0,1,\ldots,m-1 \ldots \tag{4}\)

where \(u_{i,j} = u(x_{i}, t_{j})\), \(c_{i,j} = c_{\alpha}(x_{i}, y_{j})\),
\(c_{i,j} = c_{\alpha}(x_{i}, y_{j})\) and \(s_{i,j} = s(x_{i}, t_{j})\).

The resulting equation can be explicitly solved for \(u_{i,j+1}\) to give

\[
u_{i,j+1} = \beta \left[ \sum_{k=0}^{n+1} g_{j} u_{i+k,j} + \eta \sum_{k=0}^{n+1} g_{j} u_{i+k,j} + \Delta S_{i,j} + u_{i,j} \right],
\]

\(i=1,2,\ldots,n-1, j=0,1,\ldots,m-1 \ldots \tag{5}\)

where \(\beta = c_{\alpha,i,j} \frac{\Delta t}{(\Delta x)^{\alpha}}\) and \(\eta = c_{\alpha,i,j} \frac{\Delta t}{(\Delta x)^{\alpha}}\).

Also form the initial condition and boundary conditions one can get

\(u_{i,0} = f(x_{i}), i=0,1,\ldots,n\)
\(u_{i,j} = k_{1}(t_{j}), j=0,1,\ldots,m\)
\(u_{R,j} = k_{2}(t_{j}), j=0,1,\ldots,m\)

By evaluating eq.(5) at each \(i=1,2,\ldots,n-1\) and \(j=0,1,\ldots,m-1\) and using the above three one equations one can get the numerical solutions of eq.(1).

**Finite Difference Method for Solving the Two-Sided Fractional Partial Differential Equations with Parameter**

In this section, we demonstrate the finite difference method to solve the two-sided fractional partial differential equations of the form:

\[
\frac{\partial u(x,t)}{\partial t} = \left[ c_{\alpha}(x,t) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + c_{\alpha}(x,t) \frac{\partial^\alpha u(x,t)}{\partial x^{\alpha}} \right] \ldots \tag{6}
\]

subject to the initial condition
\(u(x,0) = f(x), L < x < R \ldots \tag{7}\)

and the boundary conditions

\(u(L,t) = k_{1}(t), \ldots \tag{8}\)
\(u(R,t) = k_{2}(t), \quad 0 \leq t \leq T \)

Where \(c_{\alpha}\) and \(c_{\alpha}\) are known functions of \(x\) and \(t\), \(f\) is a known function of \(x\), \(g\) and \(k\) are known function of \(t\), \(\alpha\) is a given fractional order and \(\lambda\) is a scalar parameter. The problem here is to find the eigenpair \((\lambda,u)\) which satisfy eq.(7)-(8).

This equation can be written as an eigenvalue problem \(Au = \lambda Bu\), where

\[A = \frac{\partial}{\partial t}, B = c_{\alpha}(x,t) \frac{\partial^\alpha}{\partial x^\alpha} + c_{\alpha}(x,t) \frac{\partial^\alpha}{\partial x^{\alpha}}.\]

When \(\alpha=2\) and \(c(x,t) = c_{\alpha}(x,t) + c_{\alpha}(x,t)\),

eq.(6) becomes the following classical eigenvalue problem

\[\frac{\partial u(x,t)}{\partial t} = \lambda \left[ c_{\alpha}(x,t) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \right] \]

Similarly, when \(\alpha=1\) and \(c(x,t) = c_{\alpha}(x,t) + c_{\alpha}(x,t)\),

eq.(6) reduces to the following eigenvalue problem

\[\frac{\partial u(x,t)}{\partial t} = \lambda \left[ c_{\alpha}(x,t) \frac{\partial u(x,t)}{\partial x^\alpha} \right] \]

By following the same previous steps, eq.(6) reduces to

\[\frac{u(x_{i},t_{j+1})-u(x_{i},t_{j})}{\Delta t} = \lambda \left[ c_{\alpha}(x_{i},t_{j}) \frac{\partial^\alpha u(x_{i},t_{j})}{\partial x^\alpha} \right] \]

\[+ c_{\alpha}(x_{i},t_{j}) \frac{\partial^\alpha u(x_{i},t_{j})}{\partial x^{\alpha}} \] + O(\Delta t) \ldots \tag{8}

Also use is made of the shifted Grunwald estimate to the \(\alpha\)-th fractional derivative given by eq.(3) to reduce eq.(6) as in the following form

\[\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = \lambda \left[ c_{\alpha,i,j} \frac{\partial^\alpha u(x_{i},t_{j})}{\partial x^\alpha} \right] \]

\[+ c_{\alpha,i,j} \frac{\partial^\alpha u(x_{i},t_{j})}{\partial x^{\alpha}} \] + \(\Delta S_{i,j} + s_{i,j} \ldots \tag{9}\)

where \(u_{i,j} = u(x_{i},t_{j}), c_{\alpha,i,j} = c_{\alpha}(x_{i}, y_{j})\),
\(c_{\alpha,i,j} = c_{\alpha}(x_{i}, y_{j})\) and \(g_{i} = (-1)^{j} \frac{\alpha(a-1)-\alpha(a+k)}{k!}\)

\(k=0,1,2,\ldots\)

Also from the initial condition and boundary condition one can get

\(u_{i,0} = f(x_{i}), i=0,1,\ldots,n\)
\(u_{i,j} = k_{1}(t_{j}), j=0,1,\ldots,m\)
\(u_{R,j} = k_{2}(t_{j}), j=0,1,\ldots,m\)

By evaluating eq.(9) at \(i=1,2,\ldots,n-1\) and \(j=0,1,\ldots,m-1\) one can get a system of algebraic equations which can be solved by any suitable
method to get the eigenpair 
\( (\lambda, \{u_{i,j}\}_{j=1,2,...,n-1}) \).

**Numerical Examples**

In this section, two numerical examples are presented, showing the fractional partial differential behaviors of the solution with the parameter \( \alpha \).

**Example 1:** Consider the two-sided fractional partial differential equation:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \Gamma(0.5)x^{1.5} \frac{\partial^3 u(x,t)}{\partial x^3} + \Gamma(0.5)(0.4-x)^{1.5} + 4xe^t - 0.8e^t
\]

subject to the initial condition

\[ u(x,0) = 2x, \ 0 < x < 0.4 \]

and the boundary conditions

\[ u(0,t) = 0, \ 0 \leq t \leq 0.025 \]

\[ u(0.4,t) = 0.8 e^t, \ 0 \leq t \leq 0.025 \]

This fractional partial differential equation together with the above initial and boundary condition is constructed such that the exact solution is \( u(x,t) = 2xe^t \).

The numerical solution of example (1) by using the finite difference method for \( \Delta x = 0.05 \) and \( \Delta t = 0.0025 \)

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<th>Error</th>
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</table>

**Example 2:** Consider the two-sided fractional partial differential equation:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \left[ \Gamma(0.2)x^{1.5} \frac{\partial^4 u(x,t)}{\partial x^4} + \Gamma(0.2)(0.2-x)^{1.5} \frac{\partial^4 u(x,t)}{\partial x^4} \right]
\]

subject to the initial condition

\[ u(x,0) = x+1.125, \ 0 < x < 0.2 \]

and the boundary conditions

\[ u(0,t) = e^t + 0.125, \ 0 \leq t \leq 0.005 \]

\[ u(0.2,t) = e^t + 0.325, \ 0 \leq t \leq 0.005 \]

This fractional partial differential equation together with the above initial and boundary condition is constructed such that the exact solution is \( u(x,t) = x + e^t + 0.125 \)

The numerical solution of example (2) by using the finite difference method for \( \Delta x = 0.05 \) and \( \Delta t = 0.0025 \)

<table>
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<th>Error</th>
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**References**


الخلاصة

في هذا البحث قدمنا الحل العددي للمعادلات التفاضلية الجزئية الكسرية ذات اتجاهين مع أو بدون متغير. وان خوارزمية الحل العددي لتلك المعادلات قائمة على أساس الفروق المنتهية.

أخيراً قدمنا بعض الأمثلة العددية والتي تثبت أن الطرق العددي لحل هذه المعادلات هي طرق ذات حل مؤثر فعال مقبول.