Weakly Prime and Weakly Semiprime Fuzzy Ideals

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Abstract
In this paper we introduce the notion of weakly prime (weakly semiprime) fuzzy ideals of fuzzy ring as a generalization of weakly prime (weakly semiprime) ideals. We investigate several characterizations and properties of these concepts.

Introduction
The notion of fuzzy subset of a set $S \neq \emptyset$ is a function from a set $S$ into $[0,1]$. The first one was developed by Zadeh [1]. Liu in [2] introduced the concept of a fuzzy ring. Martines [3] introduced the notion of a fuzzy ideal of fuzzy ring.

D.D. Anderson and Eric Smith [4] introduced and studied the concept of weakly prime ideals, where a proper ideal $P$ of a ring $R$ is called weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$ [4] and $P$ is called weakly semiprime ideal if for each $0 \neq x^2 \in R$ implies $x \in P$.

In this paper we introduce the notion of weakly prime (weakly semiprime) fuzzy ideals of fuzzy ring. We investigate some basic results about weakly prime and weakly semiprime fuzzy ideal of a fuzzy ring.

Throughout this paper we assume $R$ to be a commutative ring with identity $1$ and for any fuzzy ring $X$ over $R$, $X(0)=1$.

S.1 Preliminaries
This section contains some definitions and properties of fuzzy subset, fuzzy ideals and fuzzy rings, which will be used in the next sections.

Let $R$ be a commutative ring with identity. A fuzzy subset of $R$ is a function from $R$ into $[0,1]$. Let $A$ and $B$ be fuzzy subsets of $R$ we write $A \subseteq B$ if $A(x) \leq B(x)$ for all $x \in R$, [1]. The set $\{x \in R, A(x) \geq t\}$ is called a level subset of $R$, [5] and denoted by $A_t$. If $A$ and $B$ are fuzzy subsets of $R$, then $\forall t \in [0,1]$ $A=B$ iff $A_t=B_t$, [1].

Let $f$ be a function from a set $M$ into a set $N$. A fuzzy subset of $M$ and $B$ is a fuzzy subset of $N$, then

1. $f^{-1}(f(A))=A$, whenever $A$ is $f$–invariant, [6]
2. $f(f^{-1}(B))=B$, [6].

Moreover the following definitions and properties are needed later.

**Definition 1.1 [2]:**
Let $X$ be a fuzzy subset of a ring $R$, $X$ is called fuzzy ring of $R$ if $X \neq \emptyset$ and for each $x,y \in R$:
1. $X(0)=1$,
2. $X(x+y) \geq \min \{X(x), X(y)\}$,
3. $X(x+y) \geq \max \{X(x), X(y)\}$.

**Definition 1.2 [3]:**
Let $X$ be a fuzzy ring of a ring $R$, a fuzzy subset $A$ of $R$ is called a fuzzy ideal of $X$ if $A \subseteq X$
1. $A(x-y) \geq \min \{A(x), A(y)\}, \forall x, y \in R$,
2. $A(x+y) \geq \max \{A(x), A(y)\} \forall x, y \in R$.

**Proposition 1.3 [3]:**
Let $A$ and $B$ be fuzzy ideals of a ring $R$, then $(AB)_t=A_B$.
**Definition 1.5 [7]:**
A fuzzy ideal A of a fuzzy ring X is called a prime if A ≠ λR (where λR denotes the characteristic function of R such that λR(x) = 1, ∀x ∈ R) and it satisfies \( \min\{A(xy), X(x), X(y)\} \leq \max\{A(x), A(y)\} \), for all \( x, y \in R \).

**Proposition 1.6 [7]:**
Let A be a fuzzy ideal of fuzzy ring X, then A is a prime fuzzy ideal of X iff A is an ideal of X, ∅ t ∈ [0, 1].

**Remark 1.7 [8]:**
Let \( a_t \) and \( b_t \) be two fuzzy singletons of a set S. If \( a_t = b_t \), then \( a = b \) and \( t = k \), where \( t, k ∈ [0, 1] \).

**Definition 1.8 [9]:**
Let A be a fuzzy ideal of a ring R is called a maximal ideal if:
1- A is not constant
2- For any fuzzy ideal B of R if A ⊆ B then either A = B or B = λR.

### S.2 Weakly Prime Fuzzy Ideals

In this section, we introduce the notion of weakly prime fuzzy ideal of fuzzy ring as a generalization of (ordinary) notion weakly prime ideal of a ring, where an ideal I of a ring R is called weakly prime ideal if for each \( a, b ∈ R \) such that \( 0 ≠ a·b ∈ I \), then \( a ∈ I \) or \( b ∈ I \) [4]. We shall give some properties of this concept.

**Definition 2.1:**
Let X be a fuzzy ring of a ring R. Let A be a fuzzy ideal of X, A is called a weakly prime fuzzy ideal of X if for each \( a_t, b_s ∈ X \), \( t, s ∈ (0, 1] \), such that \( O_{1}a_t · b_s ⊆ X \) then \( a_t ⊆ A \) or \( b_s ⊆ A \), where
\[
O_{1}(x) = \begin{cases} 
1 & x = 1 \\
0 & x ≠ 1 
\end{cases}
\]

**Proposition 2.2:**
Let A be a fuzzy ideal of a fuzzy ring X of a ring R. If A is weakly prime fuzzy ideal of X, then A is a weakly prime ideal of X, ∅ t ∈ (0, 1).
**Remark 2.5:**
The converse of proposition 2.4 is not true in general as the following example illustrate:

**Example:**
Consider the same example of remark 2.3
\[ A_t = \{ 0, 2 \} \text{ and } (A_2)^2 = 0, \text{ hence } A_2^2 = O_1. \] But A is not weakly prime fuzzy ideal.

Recall that for a fuzzy ring X, then
\[ \sqrt{O} = \bigcap \{ P : P \text{ is a prime fuzzy ideal of } X \} \]
\[ O_t \leq \bigcap \{ P_t : P_t \text{ is a prime fuzzy ideal of } X \}, \]

Hence \[ A_t \leq \bigcap \{ O_t = 0, \text{ otherwise} \} \]
\[ \bigcap \{ t = (O) \times (1) \} \]
\[ \bigcap \{ t = (O) \times (2) \} \]
\[ \bigcap \{ t = (0,1) \} \]

**Corollary 2.6:**
If A is weakly prime fuzzy ideal, then
\[ A \leq \bigcap O_t \text{ or } \bigcap O_t \leq A. \]

**Definition 2.7 [10]:**
Let A, B be two fuzzy submodule of fuzzy module X of an R-module M. The residual quotient of A and B denoted by (A:B) it is the fuzzy subset of R defined by:
\[ (A:B) = \{ r : r \text{ is fuzzy singleton of } R \} \]
\[ (A:B) = \{ r : r \text{ is fuzzy singleton of } R \} \]
\[ \text{sup} \{ r : r \text{ is fuzzy singleton of } R \} \]

We next give characteristics of weakly prime fuzzy ideals.

**Theorem 2.8:**
Let A be a fuzzy ideal of a fuzzy ring X of R. Then the following are equivalent:
1. A is weakly prime fuzzy ideal of X.
2. For each \( a_i, x_i \in X - A \), (A:x_i) = A \cup \{ 0 : x_i \}
3. For \( x_i \in X - A \), (A:x_i) = A or (A:x_i) = \{ 0 : x_i \}
4. For fuzzy ideals B and C of X with \( O_1 \neq BC \subseteq A \), either B \subseteq A or C \subseteq A.

**Proof:**
(1 \implies 2) Let \( y_i \in (A : x_i) \), where \( x_i \in X - A \). Then \( y_i \subseteq (O_1 : x_i) \). If \( y_i \neq x_i \), then \( y_i \subseteq (O_1 : x_i) \). If \( O_1 \neq x_i \), then \( x_i \subseteq (O_1 : x_i) \), since A is weakly prime fuzzy ideal (i.e.) \( y_i \subseteq A \) or \( A : x_i \subseteq A \cup (O_1 : x_i) \). On the other hand, \( A \subseteq (A : x_i) \). Hence \( A \cup (O_1 : x_i) \subseteq (A : x_i) \).

(2 \implies 3) it is clear, so it omitted
(3 \implies 1) clear
(1 \implies 4) If \( O_1 \neq BC \subseteq A \), then for all \( t \in (0,1) \), \( (O_1) \neq (BC) \subseteq A \). Hence \( O_t \neq BC \subseteq A \), for all \( t \in (0,1) \).

But A is weakly prime fuzzy ideal, so A_t is weakly prime ideal (by prop. 2.1). Hence \( B_t \subseteq A_t \) or \( C_t \subseteq A_t \). Thus \( B \subseteq A \) or \( C \subseteq A \).

(4 \implies 1) Suppose \( x_i, y_i \subseteq X \) such that \( O_1 \neq x_i \cdot y_i \subseteq A \), then \( O_1 \neq x_i \cdot y_i \subseteq A \) or \( y_i \subseteq A \). Hence \( x_i \subseteq A \) or \( y_i \subseteq A \). Thus A is weakly prime fuzzy ideal.

**Proposition 2.9:**
Let A be a weakly prime fuzzy ideal of X such that is not prime, then \( A \sqrt{O_t} = O_t \).

**Proof:**
Since A is weakly prime fuzzy ideal, then A_t is weakly prime ideal of X, for all \( t \in (0,1) \). Hence \( A_t \sqrt{O_t} = 0 \) by [4, Th.4], which implies \( A_t \sqrt{O_t} = (A \sqrt{O_t}) = (O_t) = O \). Thus \( A \sqrt{O_t} = O_t \).

**Corollary 2.10:**
If A and B are weakly prime fuzzy ideal of X such that are not prime, then \( AB = O_1 \).

**Proof:**
Since A and B are weakly prime, then A_t and B_t are weakly, for all \( t \in (0,1) \). Hence \( A_t \cdot B_t = O \) by [4, Th.4], so \( A(B) = O \). Thus \( AB = O_1 \).

For the following result, we need the following lemma.

**Lemma 2.11:**
Let X be a fuzzy ring of R such that \( X(a) = 1 \), for all \( a \in R \). If X is a fuzzy local ring (with unique fuzzy maximal ideal) then X is a local ring.

**Proof:**
Let \( A_1 \) and \( A_2 \) be two maximal ideals of X, now define \( B_1 : R \longrightarrow [0,1] \) and \( B_2 : R \longrightarrow [0,1] \) by:
\[ B_1(x) = \begin{cases} 1 & \text{if } x \in A_1 \\ 0 & \text{otherwise} \end{cases} \]
\[ B_2(x) = \begin{cases} 1 & \text{if } x \in A_2 \\ 0 & \text{otherwise} \end{cases} \]

It is clear that \( B_1 \) and \( B_2 \) are fuzzy ideals of X and \( (B_1)_t = A_1 \) and \( (B_2)_t = A_2 \). To prove \( B_1 \) and \( B_2 \) are fuzzy maximal ideals of X. If \( B_1 \subseteq C \subseteq X \). Hence \( (B_1)_t \subseteq C_t \subseteq X_t \), where C is a fuzzy ideal of X, which implies \( (B_1)_t = C_t \).
or $C \subseteq X$. But $C \subseteq X$ is impossible, hence $(B_1)_*=C_*$. Thus $B_1=C$. Thus $B_1$ is a fuzzy maximal ideal of $X$.

Similarly, $B_2$ is a fuzzy maximal ideal of $X$. But $X$ is fuzzy local ring, so $B_1=B_2$. Thus $X$ is a local ring.

**Theorem 2.12:**

Let $X$ be a fuzzy local ring (with unique maximal fuzzy ideal $B$) such that $B^2=O_1$, then every proper ideal of $X$ is weakly prime.

**Proof:**

Since $X$ is local fuzzy ring so $X$ has a unique maximal ideal say $B$, then $X$ is local ring with unique maximal ideal say $I$. Hence $I=B_*$. Since $B^2=O_1$, then $(B^2)_*=(O_1)_*=O$. Hence $(B)_*=O$. Thus every proper ideal of $X$ is weakly prime ideal by [4, Th.8].

Recall the following definition.

**Definition 2.13** [11]:

Let $X$ be a fuzzy ring of a ring $R$, let $a_t \in X_a$ be is called an irreducible of $X$, if $a_t=b_t c_s$ for some $b_t,c_s \in X$ implies either $<a_t>=<b_t>$ or $<a_t>=<c_s>$.

**Lemma 14:**

Let $X$ be a fuzzy ring, let $a_t \subseteq X_t$ if $a_t$ is an irreducible, then $a$ is an irreducible in $X_t$, for all $t \in (0,1]$.

**Proof:**

Suppose $a=bc$ such that $b,c \in X$, so $a_t=(bc)_t$, for all $t \in [0,1]$ such that $t=\min\{k,s\}$ and $b_k \subseteq X$, $c_s \subseteq X$. Hence $a_t=b_t c_s$, but $a_t$ is an irreducible, then $<a_t>=<b_t>$ or $<a_t>=<c_s>$, suppose $<a_t>=<b_t>$, since $a=bc$, then $<a> \subseteq <b>$, let $y \in <b>$. Thus $y=rb$, $r \in R$. $y_k=(rb)_k=r_kb_k \subseteq <b_k>$, hence $y_k=a_t=d_k$, where $\lambda=\min\{t,s\}$, $d_k \subseteq X$. Hence $k=\lambda$ and $y=ad$. This implies $y \in <a>$. Thus $<a>=<b>$. Similarly, if $<a_t>=<c_s>$.

Recall that $x$ is called weakly prime element of a ring $R$ if $<x>$ is weakly prime ideal, [4].

We fuzzify this definition as let $a_t \in X_a$, is weakly prime element of $X$ if $<a_t>$ is weakly prime fuzzy ideal of $X$, where $X$ is a fuzzy ring of $R$.

**Definition 2.15** [11]:

Let $R$ be a ring with unity $1$ and $A$ be a fuzzy subring of $R$. If $1 \subseteq A$ with $r \in [0,1]$. Then $x \subseteq A$, $x_t$ is a non-zero fuzzy singleton with $t \in [0,1]$ is said to be left (right) unit fuzzy singleton in $A$ if there exists $y \subseteq A$ such that $x_t y_t=1_r$, where $r=\min\{t,s\}$. If $x_t$ is a left and right unit, then $x_t$ is called a unit in $A$. In the commutative fuzzy subring $x_t y_t=1_r$, where $r=\min\{t,s\}$.

**Proposition 2.16:**

Let $x_t$ be a nonzero nonunit of $X$, then $x_t$ is prime $\implies x_t$ is weakly prime $\implies x_t$ is irreducible.

**Proof:**

$x_t$ is prime if $<x_t>$ is a prime fuzzy ideal, hence $<x_t>$ is weakly prime fuzzy ideal, so $x_t$ is weakly element. Now, if $x_t$ is weakly prime, to prove $x_t$ is irreducible element. Suppose $O_1 \neq x_t=b_k c_s \subseteq <x_t>$ since $<x_t>$ is weakly prime fuzzy ideal, $b_k \subseteq <x_t>$ or $c_s \subseteq <x_t>$, if $b_k \subseteq <x_t>$, then $b_k \subseteq <x_t>$. But $<x_t> \subseteq <b_k>$, thus $<b_k> = <x_t>$. Similarly, if $c_s \subseteq <x_t>$, Thus $<c_s> \subseteq <x_t>$. Thus $x_t$ is an irreducible element.

Now we can give the following result.

**Theorem 2.17:**

Let $A$ be a proper fuzzy ideal of a fuzzy ring $X$. If every nonzero element of $A$ is an irreducible, then $A$ is weakly prime.

**Proof:**

To prove $A$ is weakly prime, let $a_t, b_k \subseteq X$ such that $O_1 \neq a_t \cdot b_k \subseteq A$, so $a_t b_k$ is an irreducible. Hence $<a_t b_k>=<a_t>$ or $<a_t b_k>=<b_k>$, hence $a_t \subseteq A b_k \subseteq A$ or $b_k \subseteq a_t b_k \subseteq A$. Then $a_t \subseteq A$ or $b_k \subseteq A$. Thus $A$ is weakly prime fuzzy ideal.

**S.3 Weakly Semiprime Fuzzy Ideals**

In this section, we introduce the notion of weakly semiprime ideal of fuzzy ring as a generalization of (ordinary) notion weakly semiprime ideal, where an ideal I is called weakly semiprime ideal of $R$ if for any $x \in R$ such that $0 \neq x^2 \in I$ implies $x \in A$.

We shall give main properties of this concept.
Definition 3.1: 
Let A be a non constant fuzzy ideal of fuzzy ring X of R, A is called a weakly semiprime fuzzy ideal of X if for any fuzzy singleton \( x_t \) of R, \( O_t \neq x_t^2 \subseteq A \) implies \( x_t \in A \).

Remark 3.2: 
Every weakly prime fuzzy ideal is weakly semiprime, but the converse is not true in general for the following example shows:

Example: 
Let \( X : Z \longrightarrow [0,1] \) defined by: \( X(a) = 1 \) for all \( a \in Z \), let \( A : Z \longrightarrow [0,1] \) defined by 
\[
A(x) = \begin{cases} 
1 & x \in 6Z, \\
0 & \text{otherwise}
\end{cases}
\]
It is easy to show A is a fuzzy ideal of fuzzy ring X, \( A = 6Z \) is semiprime ideal since \( \sqrt{6} = (6) \), hence A is weakly semiprime. But A is not weakly prime because if A is weakly prime, then \( A_1 \) is weakly prime, however \( A_1 = 6Z \) is not weakly prime since \( 6 \neq 2 \cdot 6 \in A_1 \), such that \( 2 \not\in A_1 \) and \( 3 \not\in A_1 \).

Proposition 3.3 
Let X be a fuzzy ring of R and A is a fuzzy ideal of X, then A is weakly semiprime of X iff \( A_1 \) is weakly semiprime ideal of \( X_t \) for all \( t \in (0,1] \).

Proof: 
Let \( O \neq a^2 \in A_1 \), \( a \in X_t \), hence \( O \neq a^2 \subseteq A \). But A is weakly semiprime of X, so \( O \neq a \in A \). Hence \( a \in A_1 \). Thus \( A_1 \) is weakly semiprime ideal.

Conversely, to prove A is a weakly semiprime fuzzy ideal of X, let \( O_t \neq a^2 \subseteq A \), hence \( O \neq a^2 \in A_1 \). Thus \( a \in A \) since \( A_1 \) is weakly semiprime. So \( a \in A \). Thus A is weakly semiprime fuzzy ideal.

Proposition 3.4: 
If A is weakly semiprime fuzzy ideal of fuzzy ring X, then \( A = \sqrt{A} \) or \( A \subseteq \sqrt{O_t} \).

Proof: 
Since A is weakly semiprime, then \( A_t \) is weakly semiprime ideal of \( X_t \) for all \( t \in (0,1] \), by prop.3.3. Hence \( A_t = \sqrt{A} \) or \( A_t = \sqrt{O_t} \). It is clear that \( A_t = (\sqrt{A})_t \), which implies \( A = \sqrt{A} \). If \( A_t = \sqrt{O_t} = L(R) \). Thus \( A = \sqrt{O_t} \).

Proposition 3.5: 
Let A be a fuzzy ideal of fuzzy ring X. Then A is weakly semiprime fuzzy ideal iff \( A(x^2) > 0 \) implies \( A(x) = A(x^2), x \in R \).

Proof: 
To prove \( A(x) = A(x^2) \), let \( A(x^2) = t, t > 0 \), hence \( x^2 \in A_t \). Since A is weakly semiprime fuzzy ideal, then \( x \in A_t \). Thus \( A(x) \geq t \), but \( t = A(x^2) = A(x) \). Hence \( A(x) = t \), so \( A(x^2) = A(x) \).

Conversely, to prove A is weakly semiprime, let \( O_t \neq x_t^2 \subseteq A \), \( t > 0 \), hence \( A(x^2) \geq t \). But \( A(x^2) = A(x) \geq t \), so \( A(x) \geq t \), hence \( O_t \neq x_t \in A \). Thus A is weakly semiprime fuzzy ideal.

Now, we shall study the behaviour of weakly semiprime fuzzy ideal under homomorphisms.

Proposition 3.6: 
Let \( f : R_1 \longrightarrow R_2 \) be a ring epimorphism, let X and Y be two fuzzy rings of \( R_1 \), \( R_2 \) respectively and let A, B be fuzzy ideals of X, Y respectively, then
1- If A is weakly semiprime of X, then \( f(A) \) is weakly semiprime of Y. A is invariant.
2- If B is weakly semiprime fuzzy ideal of Y, then \( f^{-1}(B) \) is weakly semiprime of X.

Proof (1): 
If \( (f(A))(y) > 0 \), for each \( y \in R_2 \), \( y = f(x) \), for some \( x \in R \) since f is onto.

Moreover \( y \neq 0 \), so \( x \neq 0 \).

\[
\begin{align*}
\text{f(A)}(f(x))^2 &= f(A)(f(x^2)) \\
&= f^{-1}(f(A)(x^2)) \\
&= A(x^2) \quad \text{since f is f-invariant} \\
&= A(x), \text{since A weakly semiprime} \\
&= f^{-1}(f(A)) \\
&= f(A) f(x).
\end{align*}
\]

Thus \( f(A) \) is weakly semiprime fuzzy ideal of Y.

Proof (2): 
To prove \( f^{-1}(B)(x^2) = f^{-1}(B(x)) \), \( x \in R \), if \( f^{-1}(B)(x^2) > 0 \), hence
\[
\begin{align*}
f^{-1}(B)(x^2) &= B(f(x^2)) \\
&= B(f(x))^2, \quad \text{since f homo.} \\
&= B(f(x)), \text{since B is weakly semiprime} \\
&= f^{-1}(B(x)).
\end{align*}
\]

Thus \( f^{-1}(B) \) is weakly semiprime fuzzy ideal of X.
References


الخلاصة

في هذا البحث قدمنا فهوم المثاليات الأولية الضعيفة (شبه الأولية الضعيفة) الضبابية في حلقة ضبابية كتعقيم لمفهوم المثاليات الأولية الضعيفة (شبه الأولية الضعيفة). ثم أعطينا العديد من التشخيصات والخواص الأساسية لهذا المفهوم.