On Primary Ideal and Semi Primary Ideal of a Near Ring

المثالية الابتدائية والمثالية شبه الابتدائية في الحلقة القريبة

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Abstract

In this paper, we give some properties of primary and semi primary ideals of a near ring and link between them. Also, we determine the relationship among these concepts and other types of prime ideals of a near ring.

الخلاصة

اعطينا في هذه االبحث، بعض خواص المثالية الابتدائية و شبه الابتدائية في الحلقة القريبة وربطنا بينهما. كذلك حددنا العلاقات بين هذين المفهومين والأنواع الأخرى من المثاليات الاولية في الحلقة القريبة .

1. INTRODUCTION

In 1905 L.E Dickson began the study of near ring and later 1930 Wieland has investigated it .Further, material about a near ring can be found [3]. In 1962, R.W. Gilmer introduced the concept of Rings in which Semi Primary Ideals are Primary[6]. In 1977 G. Pilz, was introduced the notion of prime ideals of a near ring [3]. In 1988 N.J.Groenewald was introduced the notions of completely (semi) prime ideals of a near ring [7]. In 2008 G. L. Booth, K. Mogae was introducing the notions of equiprime Ideal near rings [2]. In 2011, H.H.abbass, S.M.Ibrahem introduced the concepts of a completely semi prime ideal with respect to an element of a near ring and the completely semi prime ideals near ring with respect to an element of a near ring [4]. In 2012, H.H.Abbass and M.A. Mohammed gave the notion of a completely prime ideal with respect to an element of a near ring and the completely prime ideals near ring with respect to an element of a near ring [5]. In this paper we study primary and semi primary ideals of a near ring. We prove some theorems and give some examples to show the relationships among them and other types of prime ideals.

2. PRELIMINARIES

In this section we review some basic definitions and notations of near ring, (ideal, a completely prime ideal, a primary ideal and a semiprimary ideal) of a near ring with some theorems and Propositions.

Definition (2.1):[3]

A left near ring is a set N together with two binary operations" +" and "." such that
(1) (N,+) is a group (not necessarily abelian).
(2) (N, .) is a semi group .
(3) $n_1 .(n_2 + n_3) = n_1 . n_2 + n_1 . n_3$, for all $n_1, n_2, n_3, \in N$.

Definition (2.2):[9]

Let N be a near ring. If a . b = b . a for all a, b ∈ N we say N is a commutative near ring .

Definition (2.3) :[9]

Let (N,+,.) be a near-ring. A normal subgroup I of N, is called a left ideal of N if
(1) I.N ⊆ I.
(2) ∀ n, n_1 ∈N and for all i ∈ I , $n.(n_1 + i) – n.n_1 \in I$.

Remark (2.4):
We will refer that all near rings and ideals are left in this paper.

Definition (2.5) :[7]
Let I be an ideal of a near ring N. Then I is called a completely prime ideal of N if $\forall x,y\in N$, $x.y\in I$ implies $x\in I$ or $y\in I$, denoted by C.P.I of N.

Definition (2.6):[5]
Let N be a near ring, I be an ideal of N and $x\in N$. Then I is called a completely prime ideal with respect to an element x denoted by (x-C.P.I) or(x- completely prime ideal)of N if for all $y,z\in N$, $x.(y.z)\in I$ implies $y\in I$ or $z\in I$.

Definition (2.7) :[7]
An ideal I of a near ring N is called a completely semiprime ideal (C.S.P.I) of a near ring N , if $y^2 \in I$ implies $y\in I$ for all $y\in N$.

Proposition (2.8)[5]:
Every C.P.I of a near ring N is a C.S.P.I of N.

Remark(2.9):[9]
Every ring is a near ring.

Proposition (2.10): [5]
Let N be a near ring and $x\in N$. Then every completely prime ideal with respect to an element x (x-C.P.I)of N is a completely semiprime ideal with respect to an element x (x-C.S.P.I).

Definition (2.11):[4]
Let N be a near ring and $x\in N$. Then I is called a completely semiprime ideal with respect to an element x denoted by (x-C.S.P.I) or (x- completely semiprime ideal)of N if for all $y\in N$, $x.y^2 \in I$ implies $y\in I$.

Definition (2.12):[1]
Let I be an ideal of a ring R. Then I is called a primary ideal of R if $\forall x,y\in R$, $x.y\in I$ implies $x\in I$ or $y^m\in I$ for some $m\in Z^+$, denoted by Pr.I of R.

Now, we generalize this concept to a near ring .

Definition (2.13)
Let I be an ideal of a near ring N. Then I is called a primary ideal of N if $\forall x,y\in N$, $x.y\in I$ implies $x\in I$ or $y^m\in I$ for some $m\in Z^+$, denoted by Pr.I of N.

Definition (2.14) :[2]
An ideal I of a near-ring N is called equiprime ideal, if $a \in N-I$ and $x, y \in N$ such that $anx - any \in I$ for all $n \in N$, then $x - y \in I$.

Definition(2.15) [2]:
Let I be an ideal of N. Then I is called a completely equiprime ideal of N if $a, x, y\in N$ with $ax - ay\in I$ implies $x - y\in I$ or $a\in \bar{I}$, where $\bar{I}$ is the largest ideal of N contained in I.

Corollary (2.16):[2]
Let I be a completely equiprime ideal of near ring N. Then I is an equiprime ideal of N.

Definition(2.17) [2]:

Let N be near ring and I be a proper ideal of N. Then I is called a strongly equiprime ideal of N if for each $a \in N-I$, there exists a finite subset F of N such that $x, y \in N$ and $afx - afy \in I$, for all $f \in F$, then $x - y \in I$.

Remark(2.18):[2]

If I is a strongly equiprime ideal of near ring N, then I is an equiprime ideal of N.

Proposition (2.19): [5]

Let I be an equiprime ideal of near ring N. Then I is an x-C.P.I of N , for all $x \in N - I$.

Corollary (2.20)[5]:

Every equiprime ideal of a near ring N is an x-C.S.P.I of N, for all $x \in N - I$.

Definition(2.21) [8]:

Let N be near ring and I be a subset of N. We write, radical of I, and denoted by
$\sqrt{I} = \{x \in N : x^n \in I \text{ for some } n \in Z^+\}$.

Definition (2.22):[6]

Let I be an ideal of a ring R. Then I is called a semiprimary ideal of R, if $\forall x,y \in R$, $x.y \in I$ implies $x^m \in I$ or $y^m \in I$, for some $m \in Z^+$, denoted by semi Pr.I of R.

Now, we generalize this concept to a near ring .

Definition (2.23):

Let I be an ideal of a near ring N. Then I is called a semiprimary ideal of N, if $\forall x,y \in N$, $x.y \in I$ implies $x^m \in I$ or $y^m \in I$, for some $m \in Z^+$, denoted by semi Pr.I of N.

3. THE MAIN RESULTS

Proposition (3.1):

Let N be a near ring, and I is a completely prime ideal of N. Then I is a primary ideal of N.

Proof:

$x,y \in N$ such that $x.y \in I$

$x.y \in I \Rightarrow x \in I$ or $y \in I$ [Since I is C.P.I by definition (2.5)]

$\Rightarrow$ I is a Pr.I of N. [By definition (2.13)] ▪

Theorem (3.2):

Let N be a near ring, I is an ideal of N such that $\sqrt{I} = I$. Then I is a completely prime ideal of N if and only if it is a primary ideal.

Proof: $\longrightarrow$

Is directly from the Proposition (3.1)

Conversely

Let $x,y \in N$ such that $x.y \in I$

$x.y \in I \Rightarrow x \in I$ or $y^m \in I$, for some $m \in Z^+$[Since I is a Pr.I by Definition (2.13)]

$x \in I$ or $y \in \sqrt{I}$

$x \in I$ or $y \in I$ $[\text{Since } \sqrt{I} = I]$

$\Rightarrow$ I is a C.P.I of N. [By Definition (2.5)] ▪

Corollary (3.3):

Let N be a near ring, I is a primary ideal of N such that $\sqrt{I} = I$. Then I is a completely semiprime ideal of N.

Proof:

Let I is a Pr.I of near ring N.

$\Rightarrow$ I is a C.P.I of a near ring N. [By Theorem (3.2)].

$\Rightarrow$ I is a C.S P.I of a near ring N. [By Proposition (2.8)] ▪

Proposition (3.4):

Let N be a near ring and I is a completely prime ideal of N. Then I is an x-C.P.I of N for all $x \in N - I$.

Proof:

Let $y, z \in N$, $x \in N - I$ such that $x.(y.z) \in I$

Since $x \notin I$

$\Rightarrow$ $y.z \in I$ implies $y \in I$ or $z \in I$ [Since I is C.P.I by Definition (2.5)].

$\Rightarrow$ I is an x-C.P.I of N for all $x \in N - I$. ▪

Corollary (3.5):

Let N be a near ring and I is a completely prime ideal of N. Then I is an x-C.S P.I of N for all $x \in N - I$.

Proof:

Is directly from the Proposition (3.4) and Proposition (2.11). ▪

Proposition (3.6):

Let I be a completely equiprime ideal of a near ring N. Then I is an x- C.P.I of N for all $x \in N - I$.

Proof:

Let I be a completely equiprime ideal of N

$\Rightarrow$I be an equiprime ideal of N [By Corollary (2.16)]

$\Rightarrow$ I is an x-C.P.I , for all $x \in N - I$. [By Proposition (2.19)] ▪

Corollary (3.7):

Let I be a completely equiprime ideal of a near ring N. Then I is an x- C.S.P.I of N for all $x \in N - I$.

Proof:

Let I be a completely equiprime ideal of N

$\Rightarrow$I be an equiprime ideal of N [By Corollary (2.16)]

$\Rightarrow$ I is an x-C.S. P.I , for all $x \in N - I$. [By Corollary (2.20)] ▪

Proposition (3.8):

Let I be a strongly equiprime ideal of a near ring N. Then I is an x-C. P.I of N for all $x \in N - I$.

Proof:

Let I is a strongly equiprime ideal of N.

$\Rightarrow$I be an equiprime ideal of N [By Remark (2.18)]

$\Rightarrow$ I is an x-C.P.I , for all $x \in N - I$. [By Proposition (2.19)] ▪

Proposition (3.9):

Let N be a commutative near ring and I is a completely prime ideal of N. Then $\sqrt{I}$ is completely prime ideal of N .

Proof:

Let $x,y \in N$ such that $x.y \in \sqrt{I}$

$(x.y)^n = x^n . y^n \in I$, for some $n \in Z^+$ [Since N is a commutative near ring]

$\Rightarrow x^n \in I$ or $y^n \in I$, for some $n \in Z^+$ [Since I is C.P.I by Definition (2.5)]

$\Rightarrow x \in \sqrt{I}$ or $y \in \sqrt{I}$
$\Rightarrow \sqrt{I}$ a completely prime ideal of N. ▪

Remark (3.10):

The Converse of the proposition (3.9) may be not true as in the following example.

Example (3.11):

Consider the near ring N = {0,1,2,3 } with addition and multiplication defined by the following tables .

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	0	0
3	0	1	0	1

The ideal I={0,1}is not a completely prime ideal since $3.2=0 \in I$, since $3 \notin I$ or $2 \notin I$, but $\sqrt{I}$ ={ 0,1,2,3} is completely prime ideal since $3.2=0 \in \sqrt{I}$, $2 \in \sqrt{I}$.

Proposition (3.12):

Let N be a near ring and I is a completely semi prime ideal of N. Then $\sqrt{I}$ is completely semi prime ideal of N .

Proof:

Let $y \in N$ such that $y^2 \in \sqrt{I}$
$\Rightarrow (y^2)^n = (y^n)^2 \in I$ for some $n \in Z^+$
$\Rightarrow$ $y^n \in I$, for some $n \in Z^+$ [Since I is C.S.P.I of N]
$\Rightarrow$ $y \in \sqrt{I}$
$\Rightarrow \sqrt{I}$ is an x- C.S.P.I of N. ▪

Remark (3.13):

The Converse of the proposition (3.12) may be not true as in the following example.

Example (3.14):

Consider the near ring N = {0,1,2,3 } with addition and multiplication defined by the following tables .

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	2	0	2
3	0	3	0	3

The ideal I={0}is not a completely semiprime ideal Since $2^2=0 \in I$ $2 \notin I$, but $\sqrt{I}$ ={ 0,2} is completely semi prime ideal Since $2^2=0 \in \sqrt{I}$, $2 \in \sqrt{I}$.

Proposition (3.15):

Let N be a commutative near ring and I is a primary ideal of N. Then $\sqrt{I}$ is primary ideal of N .

Proof:

Let x,y∈ N such that x.y ∈ $\sqrt{I}$

⇒ $(x.y)^n = x^n.\ y^n \in I$, for some n∈ Z^+ [Since N is a commutative near ring]

⇒ $x^n \in I$ or $(y^n)^m \in I$, for some n , m ∈ Z^+ [Since I is a Pr.I of N]

⇒ $x^n \in I$ or $(y^m)^n \in I$, for some n, m∈ Z^+

⇒ x ∈$\sqrt{I}$ or $y^m \in \sqrt{I}$, for some m∈ Z^+

⇒$\sqrt{I}$ a primary ideal of N. ▪

Remark (3.16):

The Converse of the proposition (3.15) may be not true as in the following example.

Example (3.17):

Consider the near ring N in example (3.11) the ideal I={0} is not a primary ideal since 2.3=0 ∈ I 2∉ I or $3^m \notin I$ ∀m ∈ Z^+but $\sqrt{I}$ ={ 0,2} is a primary ideal since 2.3=0 ∈ $\sqrt{I}$, 2∈ $\sqrt{I}$.

Theorem (3.18) :

Let N be a near ring and I is an ideal of N. Then $\sqrt{I}$ is a primary ideal of N if and only if it is completely prime ideal of N.

Proof: →

Let $\sqrt{I}$ be a Pr.I of N and x,y∈ N such that

x.y ∈ $\sqrt{I}$

⇒ x ∈$\sqrt{I}$ or $y^m \in \sqrt{I}$,for some m∈Z^+[Since $\sqrt{I}$ is Pr.I of N]

⇒ x ∈$\sqrt{I}$ or $(y^m)^n = y^{mn} \in I$, for some m, n∈Z^+

⇒ x ∈$\sqrt{I}$ or y∈ $\sqrt{I}$, for some m∈Z^+

⇒$\sqrt{I}$ is an C.P.I of N.

Conversely

Let $\sqrt{I}$ be a C.P.I of N then$\sqrt{I}$ is a Pr.I of N by Proposition (3.1) . ▪

Corollary (3.19):

Let N be a commutative near ring and I is an ideal of N and $\sqrt{I}$ is a primary ideal of N. Then $\sqrt{I}$ is completely semiprime ideal of N.

Proof:

It is directly by Theorem (3.18)

Proposition (3.20):

Let N be a commutative near ring, x∈ N and I is an x^n- C.P.I of N, ∀n∈Z^+. Then $\sqrt{I}$ is an x-C.P.I of N.

Proof:

Let y,z∈ N such that x.y.z ∈ $\sqrt{I}$

⇒ $(x.y.z)^n = x^n.y^n.z^n \in I$

] ⇒ $y^n \in I$ or $z^n \in I$ [Since I is x^n- C.P.I of N

⇒ y ∈$\sqrt{I}$ or z∈ $\sqrt{I}$

⇒$\sqrt{I}$ is an x- C.P.I of N. ▪

Proposition (3.21):

Let N be a commutative near ring, x∈ N and I is an x^n- C.S.P.I of N, ∀n∈Z^+. Then $\sqrt{I}$ is an x-C.P.I of N.

Proof:

Let $y \in N$ such that $x.y^2 \in \sqrt{I}$

$\Rightarrow (x.y^2)^n = x^n.(y^n)^2 \in I$, for some $n \in Z^+$

$\Rightarrow y^n \in I$ for some $n \in Z^+$ [Since I is x^n- C.S.P.I of N for each $n \in Z^+$]

$\Rightarrow y \in \sqrt{I}$

$\Rightarrow \sqrt{I}$ is an x- C.S.P.I of N. ▪

Remark (3.22):

Its clear that every primary ideal of a near ring N is semiprimary ideal of N. ▪

Theorem (3.23):

Let N be a near ring, I is an ideal of N such that $\sqrt{I} = I$. Then I is primary ideal of N if and only if it is semi primary ideal of N.

Proof:

If I primary ideal of N, then by Remark(3.31) it is semi primary ideal of N.

Conversely

Let $x,y \in N$ such that $x.y \in I$.

$\Rightarrow x^m \in I$ or $y^m \in I$, for some $m \in Z^+$. [Since I is semi Pr.I by definition (2.23)]

$\Rightarrow x \in \sqrt{I}$ or $y^m \in I$ for some $m \in Z^+$.

$\Rightarrow x \in I$ or $y^m \in I$, for some $m \in Z^+$. [since $\sqrt{I} = I$]

$\Rightarrow$ I is a primary ideal of N. ▪

Proposition (3.24):

Let N be a near ring, and I is a completely prime ideal of N. Then I semiprimary ideal of N.

Proof:

Is directly from the Proposition (3.23) and Remark (3.22). ▪

Theorem (3.25):

Let N be a commutative near ring, and I is an ideal of N. Then I is a semiprimary ideal of N if and only if $\sqrt{I}$ semiprimary ideal of N.

Proof:

suppose I is a semiprimary ideal, and $x,y \in N$ such that $x.y \in \sqrt{I}$,this implies there exists $n \in Z^+$ such that $(x.y)^n \in I$

$\Rightarrow (x.y)^n = x^n.y^n$ [Since N is commutative near ring]

$\Rightarrow (x^m)^n \in I$ or $(y^m)^n \in I$, for some $m \in Z^+$[Since I is semi Pr.I of N]

$\Rightarrow x^m \in \sqrt{I}$ or $y^m \in \sqrt{I}$

$\Rightarrow \sqrt{I}$ is a semiprimary ideal of N

Conversely

Let $x,y \in N$ such that $x.y \in I$

$x.y \in \sqrt{I} \Rightarrow x^m \in \sqrt{I}$ or $y^m \in \sqrt{I}$ for some $m \in Z^+$ [Since $\sqrt{I}$ is semi Pr.I of N]

$\Rightarrow (x^m)^n \in I$ or $(y^m)^{\acute{n}} \in I$, for some $m, n, \acute{n} \in Z^+$ [By Definition (2.21)]

$\Rightarrow x^r \in I$ or $y^r \in I$, for some $r \in Z^+$[Since r = m.n.ń]

$\Rightarrow$ I semi primary ideal of N. ▪

Theorem (3.26):

Let N be a near ring, I is an ideal of N. Then $\sqrt{I}$ is a completely prime ideal of N if and only if I is semiprimary ideal of N.

Proof: $\longrightarrow$ suppose $\sqrt{I}$ is a completely prime ideal of N and $x,y \in N$ such that $x.y \in I$

$\Rightarrow x.y \in \sqrt{I} \Rightarrow x \in \sqrt{I}$ or $y \in \sqrt{I}$ [Since $\sqrt{I}$ is C. P.I of N]

$\Rightarrow x^m \in I$ or $y^m \in I$ for some $m \in Z^+$ [By Definition (2.21)]

$\Rightarrow$ I is a semiprimary ideal
Conversely
x,y $\in$ N such that x.y $\in\sqrt{I}$
x.y$\in$ I $\Rightarrow x^m \in$ I or $y^m \in$ I , for some m$\in Z^+$ [Since I is semiPr.I by Definition (2.23)]
$\Rightarrow$ x $\in \sqrt{I}$ or y $\in\sqrt{I}$
$\Rightarrow\sqrt{I}$ is a completely prime ideal of N. ▪

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