Group Theory And Neighborhood Building

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Abstract

There are many applications of the optimal solution problems such as the traveling salesman problem (TSP) and the location pickup and delivery problem (LPDP) , the group theory is widely used to solve such problems where the solution space of these problems become larger and larger as the nodes of the problem increase and the searching begin by constructing a neighborhood of the known solution and then lock for the optimal solution in this neighborhood and in the other neighborhood if these neighborhood are communicated , but if they are non-communicated the searching will stuck in one neighborhood and will never find the optimal solution if it was in another neighborhood . For this the searching in this case was began in 1999 see [1] . In this research we will study the constructing neighborhood using template strategy under the action of a conjugation classes of n-cycles and under the action of subgroups of S_n and their transition matrices. And using conjugation strategy under the sylow p-subgroup of S_p and discuss their transition matrices.

Keywords:- group theory - Metaheuristics - Search theory - Tabu search – optimization theory .

نظرية الزمر و انشاء الجوارات

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الملخص

ان تطبيقات مسائل الامثلية كثيرة ومنها مسألة البائع المتجول ومسالة الاخذ والتسليم وان نظرية الزمر استخدمت كثيرا في حل هكذا مسائل حيث ان فضاء الحلول لهذه المسائل كبير جدا كلما ازداد عدد الرؤوس في المسالة. وان البحث عن حل امثل ابتدءا من حل معين سيستخدم جوار لهذا الحل ومنه ننتقل الى الجوار الاخر اذا وجد ارتباط بين الجوارات ولكن ان لم يكن هنالك ارتباط فان البحث سيبقى يبحث في نفس الجوار دون جدوى ان كان الحل الامثل في جوار اخر ولهذا بدء البحث منذ عام 1999, متى تكون الجوارات مرتبطة ومتى لا تكون مرتبطة في هذا البحث سندرس انشاء جوار باستخدام مبدأ القالب تحت تأثير صفوف الاقتران وأيضا تحت تاثيرالزمر الجزئية وباستخدام مبدأ الاقتران تحت تأثير الزمر الجزئية السيلوفية من النمط P و مناقشة مصفوفات الانتقال لكل منهم .

الكلمات الدالة : نظرية الزمر – خوارزميات الاستدلال الفوقية – نظرية البحث – جدول البحث- نظرية الامثلية

Introduction

Tabu search belong to general class of optimization procedure that utilize iterative techniques to find the optimal solution. The general procedure for these iterative techniques is to construct a new solution s_j from a current solution s_i and to check whether or not procedure should stop or perform another step.

Neighborhood search methods are iterative techniques that first define a neighborhood N (s_i) for each current feasible solution s_i , and the next solution s_j is selected from among the solutions in N(s_i) . The neighborhood $N(s_i) \subset S$ represents the set of all solutions s_i' $\in N(s_i)$ that can be directly reached from the current solution s_i by a single move operation (template or conjugation).

Applications

1- LPDP: the pickup and delivery problem covers the general situation where a fleet of vehicles must serve a set transportation request. These requests specify pickup and delivery locations .Potential solutions route each vehicle to service all requests , satisfying vehicle capacity constrain while optimizing a desired object function . The cycles from the symmetric group on n letters provide a compact solution representation of the LPDP , figure(1) provide a representation of it .

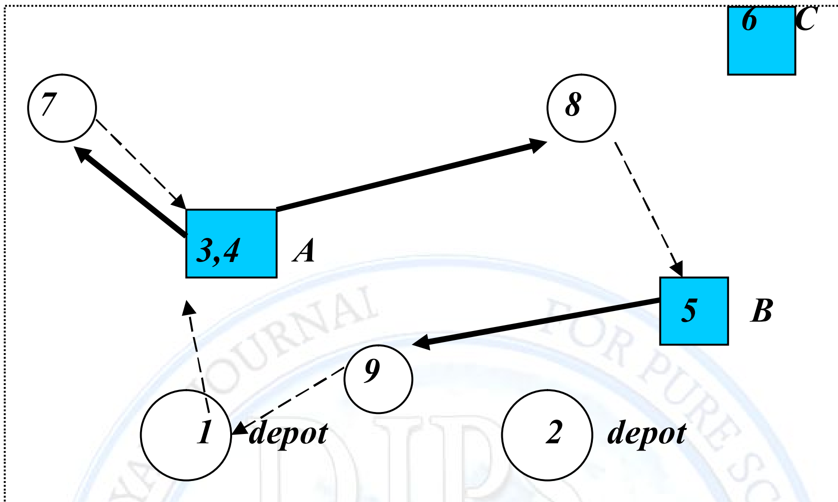


Fig. 1 graphical example of LPDP

It represent a problem consisting of two potential depot location {1,2} , that can both support the same type vehicle , three demands or delivery points {7,8,9} and three potential supply pickup points {A,B,C} group theory allows the solution structure for depicted problem and information provided to be represented in cyclic form (1 10 3 7 4 8 5 9)(10)(6)(11) each subtour represent a rout of a vehicle type it is obvious when comparing the two representations that the disjoint cycle form much easier to interpret . In this representation only one depot (1) was selected and vehicle 10 traveled to customers **3**-7-**4**-8-**5**-9 in order ,where the bold number represent pickup locations and then back to the depot .The second vehicle depot , supply pickup point C and vehicle 11 where not required to solve problem for this we use template see[6].

2- (1-TSP):

The n-city traveling salesman problem is to seek for the shortest round trip among n-cities visiting each city only one see [1], this trip can be represented by cycle of length n (1 2 3 …n) and the conjugation class of the same cycle structure are the incumbent solutions , so the search for the optimal trip , if p is a one trip then q is the other solution come from by a specific move .

§1:Fundamental concepts :

(1.1) Definition: A template is a mechanism that either fragment a permutation (splitting template) or join several smaller permutations into a single larger permutation (welding template) see[1] .

(1.2) Example : In S_8 we have (1 2 3 4 5 6 7 8) and the splitting template s = (2 5 7) create the following disjoint cycle solution:

(1 2 3 4 5 6 7 8)(2 5 7) = (2 3 4)(5 6)(7 8 1)

and the following consisting of three disjoint cycles
(1 2 3)(4 5 6)(7 8) and the welding template w = (1 4 7) create the following united cycle: (1 2 3)(4 5 6)(7 8)(1 4 7) = (1 2 3 4 5 6 7 8)

(1.3) Definition : Conjugation strategy

When the solution space is a conjugacy class in S_n the conjugation offers a way to reach other solution that is ($p^x = q$) , represents a direct move from p to q where q have the same cycle structure [6].

(1.4) Remark : Template can move between permutation (solution) with different cycle structure while conjugation assigns and rearranges letters with in partition.

(1.5) Definition : The orbit of a point in s$\in$ S is defined as the set.

$orb_G = \{s' \in S \mid s = g \oplus s$ for some g $\in$G$\}$where $\oplus$is the operation of action we use [3].

(1.6) Definition : A neighborhood is the set of all results come from the using strategy [6].

(1.7) Definition : A Transition matrix is a matrix with the following entries for all $p,q \in X$ where X is the set of all the incumbent solutions with some specific cycle structure , H is the strategy set and $\oplus$ is a binary operation(any strategy move conjucate or template).

[1]:

$$T_{pq} = \begin{cases} * .....if .....p \oplus h = q.....for..some..h \in H \\ ....otherwise \end{cases}$$

(1.8) Remark :

1. We will introduce the same example but with different actions to make the behavior of the neighborhoods under the different actions clear.

2. The transition matrices of the examples are plot after re-labeling the neighborhoods see [7]

§2:Template transition matrices under the action of conjugacy class

Consider the LPDP problem such that it's solution space X is the conjugation class of a known solution (permutation p) in S_n and we desire to find the neighborhood of p by template strategy x to find an attractive but unknown solution q ((i.e.) px=q),

first we introduce the following new definition :

(2.1) Definition: A local neighborhood is the set of the results under the using strategy with specific cycle structure (the wanted cycle structure).

(2.2) Constructing the template transition matrix under the action of conjucacy class.

To construct the transition matrix of this case we will follow the following steps:

1- Determined the cycle structure of the optimal solution q

2- Solve the equation px=q which is $x = p^{-1}q$ (it is unique).

3- Ttake the conjugacy class of S_n of the same cycle structure of x and construct the neighborhood G of p G=XC.

4- Take the local neighborhood A of p which is a subset of G.

5- Finally build the transition matrix of the solution space and the local neighborhoods and see the behavior of the neighborhood

We conclude that:

- The transition matrix is of one orbit (communicated neighborhoods) so the algorithm will proceeds without fear of stuck in the same neighbor.
- $PX = (P^{-1}X)^{-1}$.

(2.3) Example : Let p=(1234) the solution space 4-cycle permutation:

X={(1234),(1243),(1324),(1342),(1423),(1432)}if we desire q=(1324) then $x = p^{-1}q$ =(1432)(1324)=(234) so we must take the conjugacy class of three numbers which are : C={(123),(132),(124),(142),(134),(143),(234),(243)}

We construct G=XC then we gets the following neighborhoods :

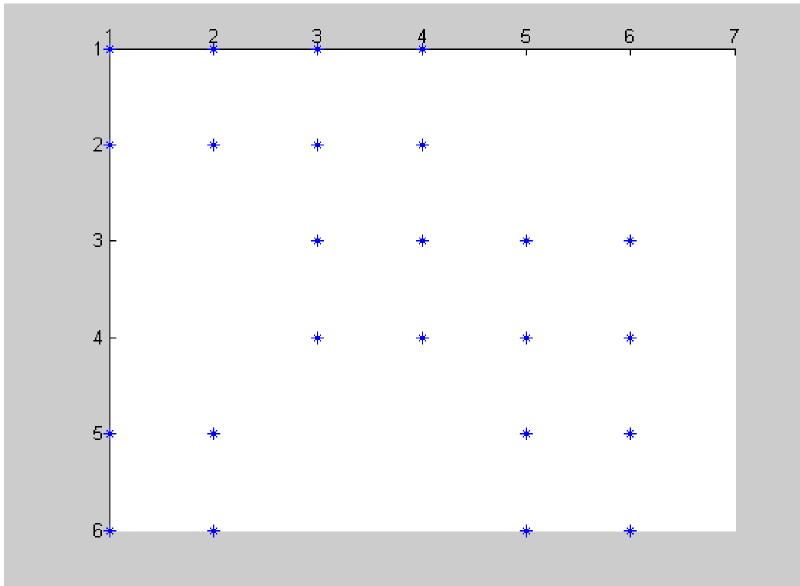
- {(1342),(34),(1423),(23),(1243),(12),(1324),(14)}
- {(1324),(24),(1432),(34),(12),(1234),(13),(1423)}
- {(24),(1243),(1342),(13),(1432),(23),(14),(1234)}

Note that here that $PX = (P^{-1}X)^{-1} = P^{-1}X$.

We take the local neighborhoods $A \subseteq G$:

{(1342),(1423),(1243),(1324)} , {(1324),(1432),(1234),(1423)} , {(1243),(1342),(1432),(1234)}.

Then we build the transition matrix:



(2.4) Example :

Let p=(12345) then $|X| = 5!/5 = 24$ and if we want q=(13452)then

$x = p^{-1}q$ =(15432)(13452)=(123) thus $|C| = 5!/3.1! = 20$ permutations of 3-cycle ,Then we have the following different neighborhoods G=XC:

- {(13452),(345),(14523),(23)(45),(15234),(234),(12453),(12)(45), (12534),(12)(34),(12354),(123),(13245),(145),(14235),(15)(23), (13425),(15)(34),(12435),(125)} thus we have the local neighborhood:

{(13452),(14523),(15234),(12453),(12534),(12354),(13245),(14235), (13425),(12435)} .

Without loss of generality we will list the local neighborhoods only :

{(13542),(14235),(15423),(12435),(12543),(12345),(13524),(13254), (15234),(12534)} .

{(13524),(14352),(15243),(12354),(12453),(12543),(14235),(13245), (14325),(12345)} .

{(13245),(14532),(15324),(12345),(12435),(12534),(14523),(15243), (14253),(12543)} .

{(13425),(14253),(15342),(12543),(12345),(12453),(13254),(15234), (15324),(12354)} .

{(13254),(14325),(15432),(12534),(12354),(12435),(14253),(15423), (15243),(12453)} .

{(12453),(13452),(13524),(14532),(15324),(13254),(12345),(12435), (13425),(14325)} .

{(12543),(13425),(13542),(14325),(15432),(13245),(12534),(12354), (13524),(15324)}.

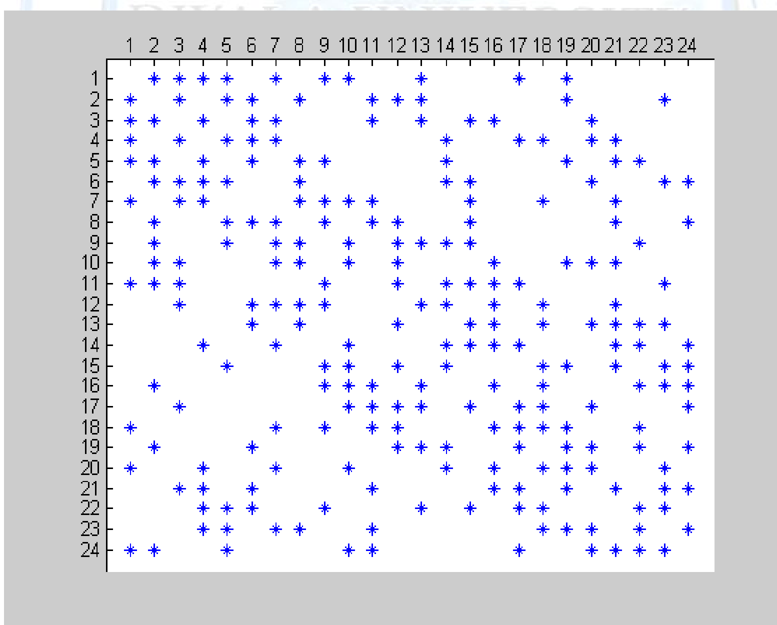
{(12534),(13254),(13452),(14253),(15342),(13542),(14325),(12345), (13245),(14235)}.

{(12435),(13542),(13245),(14352),(15243),(13452),(12354),(15324), (13254),(15234)}.

{(14352),(12354),(14523),(13542),(14253),(15423),(12435),(14325), (12345),(13425)}.

{(14253),(12543),(14352),(13254),(14532),(15432),(13425),(14235), (12435),(13245)}.

We don't need to state the reset 12 neighborhoods since we can get it from the above neighborhoods since $PX = (P^{-1}X)^{-1}$,so its transition matrix:



§ 3: Template transition matrices under the action of subgroups of the symmetric groups

In this section we will discus the case when the template strategy is used under a subgroup of S_n the final shape of transition matrix of such strategy is summarized **by the following new theorem**:

(3.1) Theorem : Let X be any solution space and H be any subgroup of S_n then the blocks of the transition matrices under template strategy by H will be non-communicated and the order of each block will be |H|.

Proof :

Let $p \in X$ thus its neighborhood under the template strategy will be $pH = \{ph | h \in H\}$ is exactly the left coset of H. Furthermore $\forall x, y \in pH$ then we have $x = ph_1$ and $y = ph_2$ for some $h_1, h_2 \in H$ thus we have xH=yH=pH thus the elements of each orbit will stuck and not communicate with other orbits .

If $p \in X \cap H$ then pH=H if $p \notin H$ then its coset have the same order of H and |X| will be divided into non-communicated block in the matrix .

(3.2) Example : Let X be the conjugate class of 5-cycle as in example (2.4) and let H be the dihedral subgroup in S_5 so |X|=24 element and H={i , (12345), (13524), (14253), (15432),(12)(35),(13)(45),(14)(23), (15)(24),(25)(34)}.

for each $x \in X \cap H \Rightarrow xH = H$ so four element of X have the same orbits :

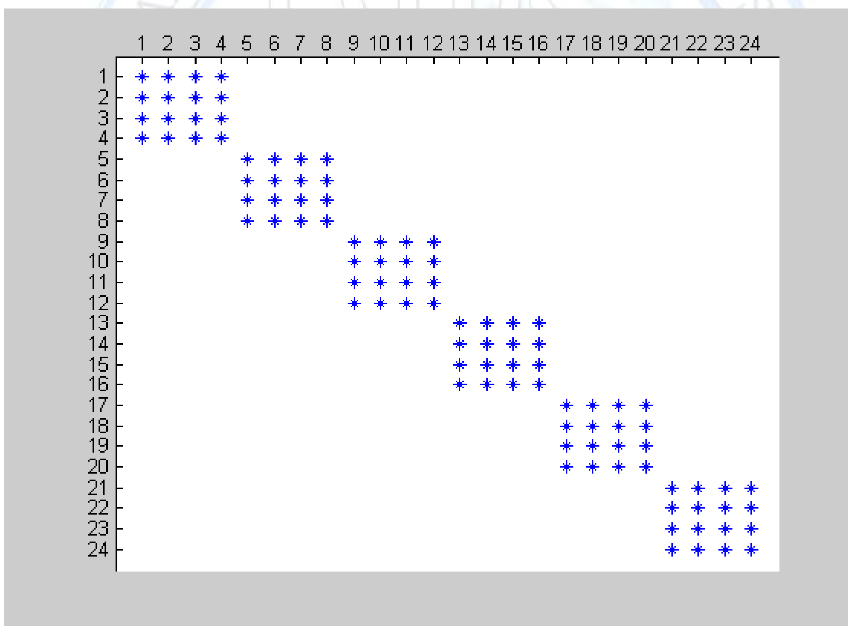
(12345)H=(13524)H=(14253)H=(15432)H=H

The rest 20 element of X will be divided into the following non-communicated orbits:

- {i,(12354),(13)(24),(14325),(152),(453),(254),(12)(34),(135),(14523), (15324)the local neighborhood is:{(12354),(14325),(14523),(15324)}.

- {(12435),(13)(25),(14532),(154),(234),(245),(12534),(13542),(143), (15)(23)} the local neighborhood: {(12435),(14532),(12534),(13542)}.
- {(12453),(13254),(142),(15)(34),(235),(243),(125),(13452),(14)(35), (15423)} the local neighborhood is: {(12453),(13254),(13452),(15423)}.
- {(12543),(132),(145),(15234),(24)(35),(23)(45),(124),(13425),(14352), (153)} the local neighborhood: {(12543),(15234),(13425),(14352)}.
- {(13245),(14)(25),(15342),(354),(123),(15243),(253),(12)(45),(134), (14235)} the local neighborhood: {(13245),(15342),(15243),(14235)}.

The transition matrix :



§4 :Conjugation Strategy Under The Action of Sylow p-subgroups

In the last two sections we study template strategy , in [2,3],[7] ,[8] they study the conjugation strategy under the action of conjugacy classes , under the action of the union of conjugacy classes and under the action of dihedral subgroups respectively .

In this section we will also study the neighborhood and the transition matrices of conjugation strategy of one traveling salesman problem (1-TSP) but under the action of sylow p-subgroups of S_n but when n=p (where p is prime number).

The following lemma refer to [4]

(4.1) Lemma: If $a \in G$ then the numbers of conjugate to a is equal to the index of its centralizer : $a^G = [G : C_G(a)]$ where $C_G(a)$ is the centralizer of a .

We will give the following new theorem :

(4.2) Theorem : If X is the solution space of a 1-TSP (p-cycle) and H is the sylow p-subgroup in S_p then the transition matrix under conjugation strategy by H will consist of $(|X| - |H-1|) \div |H|$ non-communicated different orbits.

Proof :

Since the conjugation under H is an equivalent relation [5] thus it partition the set X into equivalent classes .

Also $|S_p|$ = p!, and p is a prime number, the highest power of p that divides $|S_p|$ is p. Therefore the Sylow p-subgroups are precisely the cyclic subgroups of order p, each generated by a p-cycle. There are (p-1)! = p! / p ways to construct a p-cycle (a_1, . . . ,a_p). is the subgroup generated by p i.e. <x>=H $\Rightarrow$| H | = p $\Rightarrow H \subset X$, then since the conjugation is rearranging the elements of x and x is non-trivial cycle so $x \neq e$.

Now $\forall x (\neq e) \in X$ we have the two cases:

If $x \in H$ the centralizer of x is precisely < x >=H thus by above lemma $| x^H | = [H : C_{S_n}(x)] = [H : H] = 1$ so we have |H-1| different elements each one is of orbit have one element .

If $x \notin H$ then

$\forall y \in X^H, \exists h \in H \Leftrightarrow y = x^h = hxh^{-1} \Leftrightarrow h^{-1}y(h^{-1})^{-1} = x \Leftrightarrow x \in y^H \Rightarrow x^H = y^H$ thus the orbits are non-communicated and each orbit has | H | different elements hence the number of the blocks in the transition matrix is : $(|X| - |H - 1|) \div |H|$.

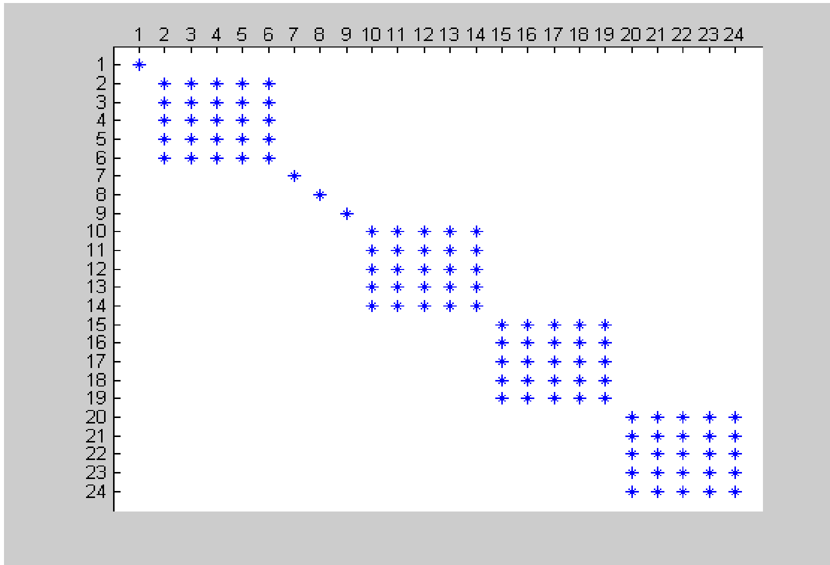
(4.3) Example : Consider the 1-TSP problem with 5-cities then the 5-cycles as in example (2.4) the sylow 5-subroup in S_5 is :

S = <(12345)> = {i,(12345),(13524),(14253),(15432)}

The neighborhoods under conjugation X^S:

- {(12345)}
- {(12354),(12435),(13245),(13452),(15234)}
- {(12453),(12534),(13425),(14235),(14523)}
- {(12543),(14325),(14532),(15342),(15423)}
- {(13254),(13542),(14352),(15243),(15324)}
- {(13524)}
- ({14253)}
- {(15432)}

And the transition matrix will be :



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