On a Completion of Fuzzy Measure

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Abstract: In this paper, we introduce some properties in completeness of fuzzy measure and we get some relations between them.

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1. Introduction

The fuzzy measure, defined on a classical $\sigma - field$, was introduced by Sugeno [7]. Ralescu and Adams [1] generalized the concepts of fuzzy measure and fuzzy integral to the case that the value of a fuzzy measure can be infinite, and to realize an approach from subjective.

Wang [12,11]and Kruse [4] studied some structural characteristics of fuzzy measures and proved several theorem about fuzzy measure.

The notion of fuzzy measure was extended by Avallone and Barbieri, Jiang and Suzuki [9], Narukawa and Murofushi[10], Ralscu and Adams [1] as a set function which was defined on $\sigma - field$ with valus in $[0, \infty]$. After that, many authors studied the fuzzy measure and proved some results about it as Guo and Zhang [10], Kui [6], Li and Yasuda [3], Lushu and Zhaohu [5], Minghu[2].

In this paper, we mention the definition of completion of fuzzy measure with some properties, and prove some new relations deal with completeness of fuzzy measure.

Definition (1):[13]

Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is called a fuzzy measure if

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leq \mu(B)$, where $A \subseteq B$

Definition (2):

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space, $A \in \mathcal{F}$ is said to be $\mu - null$ set if $\mu(A) = 0$. The fuzzy measure μ is said to be complete on $\mathcal{F}$ if $\mathcal{F}$ contains the subset of every $\mu - null$ sets.

Definition (3):[12]

μ is called countably weakly null-additive, if for any $\{A_n\} \subset \mathcal{F}$,

$$\mu(A_n) = 0 \text{ , for all n} \geq 1 \Longrightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$$

Definition (4):[12]

μ is said to be additive, if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$.

2. Main results

Theorem (1):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measurable space and μ is countably weakly null-additive and $\delta_\mu = \{E: E \subset A \in \mathcal{F}$ and $\mu(A) = 0\}$.Then δ_μ is $\sigma - ring$.

Proof:

1. Clearly $\emptyset \in \delta_\mu$.

2. Let $E_1 , E_2 \in \delta_\mu \Longrightarrow$ there exists $A_1 , A_2 \in \mathcal{F}$ such that $E_1 \subset A_1 , E_2 \subset A_2$ and $\mu(A_1) = 0 , \mu(A_2) = 0$.

$E_1 / E_2 \subset E_1 \subset A_1 \in \mathcal{F}$ So $E_1 / E_2 \in \delta_\mu$.

3. Let $\{E_n\}$ be a sequence of sets in δ_μ n=1,2,… $\Longrightarrow$ there exist a sequence $\{A_n\}$ n=1,2,… of sets in $\mathcal{F}$ such that E_n / A_n and $\mu(A_n) = 0$.

$$\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} A_n$$

Since $\mathcal{F}$ is $\sigma - field$

$$\Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$$

Since μ is countably weakly null-additive

$$\Longrightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0 .$$

So

$$\bigcup_{n=1}^{\infty} E_n \in \delta_\mu$$

Therefore

$$\delta_\mu \text{ is } \sigma - ring$$

Theorem (2):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measurable space and μ is additive, define $\bar{\mathcal{F}} = \{(E \cup E_1)/E_2 : E \in \mathcal{F} , E_1 , E_2 \in \delta_\mu\}$.Then $A \in \bar{\mathcal{F}}$ iff there exists $M, N \in \mathcal{F}$ such that $M \subset A \subset N$and $\mu(N/M) = 0$.

Proof:

Let $M, N \in \mathcal{F}$ and $M \subset A \subset N$ and $\mu(N / M) = 0$.

So

$$A = (N \cup \emptyset) / (N / A)$$

Since

$$N/A \subset N/M \in \mathcal{F} \text{ and } \mu(N/M) = 0 \Longrightarrow N/A \in \delta_\mu .$$

Therefore

$$A \in \bar{\mathcal{F}}.$$

Suppose that $A \in \bar{\mathcal{F}}$, then $= (E \cup E_1)/E_2$, $E \in \mathcal{F} , E_1 , E_2 \in \delta_\mu$.

$$\Longrightarrow \text{there exist } A_1 , A_2 \in \mathcal{F} \text{ such that } \mu(A_1) = 0 , \mu(A_2) = 0$$

and $E_1 \subset A_1 , E_2 \subset A_2 , E / A_2 \subset A \subset E \cup A_1$

$$E \cup A_1 , E/A_2 \in \mathcal{F} \text{ and } \mu((E \cup A_1)/(E/A_2))$$

$$= \mu\big((A_1 / E) \cup (A_2 \cap E)\big) = \mu(A_1 / E) + \mu(A_2 \cap E)$$

Since

$$A_1/E \subset A_1 \text{and } A_2 \cap E \subset A_2 \Longrightarrow \mu(A_1/E) = 0 \text{ and } \mu(A_2 \cap E) = 0$$

So

$$\mu((E \cup A_1)/(E/A_2)) = 0.$$

Corollary (1):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measurable space and μ is additive. Then $A \in \bar{\mathcal{F}}$ iff $A = E \cup M , E \in \mathcal{F}$ and $M \in \delta_\mu$.

Proof:

Suppose that $A \in \bar{\mathcal{F}}$. By theorem (2) there exist $M, N \in \mathcal{F}$ such that $N \subset A \subset M$ and $\mu(M \,/\, N) = 0$

$$A = N \cup (A/N) \;\; , N \in \mathcal{F}$$

Since

$$A/N \subset M/N \in \mathcal{F} \text{ and } \mu(M \,/\, N) = 0 \Longrightarrow A/N \in \delta_\mu$$

Conversely

Suppose $A = E \cup M$, $E \in \mathcal{F}$ and $M \in \delta_\mu$

$$A = (E \cup M)/\emptyset \qquad , \emptyset \in \delta_\mu \Longrightarrow A \in \bar{\mathcal{F}}$$

Corollary (2):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measurable space and μ is additive. Then $A \in \bar{\mathcal{F}}$ iff $A = E \,/\, D$ with $E \in \mathcal{F}$ and $D \in \delta_\mu$.

Proof:

Suppose that $A \in \bar{\mathcal{F}}$

$$\Longrightarrow \text{there exist } M, N \in \mathcal{F} \text{ such that}$$

$$N \subset A \subset M \text{ and } \mu(M \,/\, N) = 0$$

$$A = M/(M/A) \;\; , M \in \mathcal{F}$$

Since

$$M/A \subset M/N \in \mathcal{F} \text{ and } \mu(M \,/\, N) = 0$$

So

$$M/A \in \delta_\mu .$$

Conversely

Suppose that $A = E \,/\, D$ where $E \in \mathcal{F}$ and $D \in \delta_\mu$

$$\Longrightarrow A = (E \cup \emptyset)/D \qquad D, \emptyset \in \delta_\mu$$

$$\Longrightarrow A \in \bar{\mathcal{F}}$$

Theorem (3):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measurable space and μ is additive. Then $\bar{\mathcal{F}}$ is $\sigma - ring$.

Proof:

1. Clearly $\emptyset \in \bar{\mathcal{F}}$.

2. Let $\{A_n\}$ $n = 1,2, \ldots$ be a sequence of sets such that $A_n \in \bar{\mathcal{F}}$

$$\Longrightarrow A_n = M_n \cup N_n \text{ where } M_n \in \mathcal{F} \text{ and } N_n \in \delta_\mu .$$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (M_n \cup N_n) = \left(\bigcup_{n=1}^{\infty} M_n\right) \cup \left(\bigcup_{n=1}^{\infty} N_n\right)$$

Since

$$\mathcal{F} \text{ is } \sigma - field \text{ and } \delta_\mu \text{ is } \sigma - ring$$

$$\Longrightarrow \bigcup_{n=1}^{\infty} M_n \in \mathcal{F} \; , \bigcup_{n=1}^{\infty} N_n \in \delta_\mu$$

So

$$\bigcup_{n=1}^{\infty} A_n \in \bar{\mathcal{F}}$$

3. Let $A, B \in \bar{\mathcal{F}}$ from Corollary(1) we obtain $A = M_1 \cup N_1$

$B = M_2 \cup N_2$.

$$A/B = (M_1 \cup N_1)/(M_2 \cup N_2)$$

$$= \big((M_1 \,/\, M_2) \,/\, N_2\big) \cup \big((N_1 \,/\, M_2) \,/\, N_2\big)$$

$$= \big[\big((M_1 \,/\, M_2) \,/\, E_2\big) \cup \big((E_2 \,/\, N_2) \cap (M_1 \,/\, M_2)\big)\big] \cup \big((N_1 \,/\, M_2) \,/\, N_2\big)$$

$$N_2 \subset E_2 \in \mathcal{F} \;\; , \mu(E_2) = 0 \qquad A/B \in \bar{\mathcal{F}}$$

Therefore

$$\bar{\mathcal{F}} \text{ is } \sigma - ring.$$

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المستخلص : في هذا البحث ، قدمنا بعض الخصائص في كمالية القياس الضبابي وحصلنا على بعض العلاقات بينها