On Blow-up Solutions and Times Of a Fourth Order Nonlinear Partial Differential Equation

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ABSTRACT
In this paper, we study blow-up solutions and blow-up times of a fourth order nonlinear partial differential equation. We show that the classical solutions of this equation blow up in $C^2$, i.e. the second order derivatives blow up in $L_{\infty}$. The two steps finite difference scheme is used to compute the approximate values to the blow-up solution and times for a numerical experiment.

1. INTRODUCTION
In this paper, we study a fourth order nonlinear partial differential equation which takes the form:

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t \partial x^2} + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 = 0, \quad x \in [0,1] \quad \ldots \ldots (1)$$

with the initial and boundary conditions:

$$u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0$$

$$u(x, 0) = u_0(x),$$

where $u_0$ is a smooth function, and $u_0''(x)$ is positive function in $(0,1)$.

Blow-up phenomena for partial differential equations has been studied by many authors, see for instance [1-10]. It is well known that the classical solution of this problem can be continued in time, only if all the derivatives these appear in the equation are continuous, which means, in order to show that the solution of the problem blows up in finite time $T$ in $C^2$, it is sufficient to show that the second order derivative blows up in $L_{\infty}$.

Since it is not easy to deal with problem (1) directly, we can deal with the second order reducing problem, which can be rewritten in equation (1) as follows:

$$\frac{\partial w}{\partial t} = \frac{\partial^4 w}{\partial x^4} + w^2, \quad x \in [0,1] \quad \ldots \ldots (2)$$

where

$$w = \frac{\partial^2 u}{\partial x^2} \quad \ldots \ldots (3),$$

$$w(0, t) = w(1, t) = 0$$

$$w(x, 0) = w_0(x) = u_0''(x)$$

2. Blow-up Results
The local existence of equation (2), can be guaranteed (see [5,9]), moreover, unless $w$ is unbounded, the differential equation (3) has a unique solution, see [9]. This leads to: equation (1) has a unique local classical solution. While, it is well known that, for large initial function, the solution of problem (2) blows up in finite time at only a single point, see [5,10,11,12], which means there exist $T_{b}(b) > 0$, such that

$$\sup_{x \in [0,1]}|w(x, t)| \rightarrow \infty \quad \text{as} \quad t \rightarrow T_{b}$$

It follows that the solution of the general problem (1), has to blow-up in $C^2$-space at $T_{b}$ (the problem has no global classical solution). In fact, since the second derivative becomes unbounded in finite time, this will handle the continuity of solution, and eventually, the solution becomes unbounded in equal or larger finite time.

From above, we see that, in order to find the solution of problem (1), we can first solve problem (2) and then we...
substitute $w$ in equation (3), and finally from solving equation(3), we get the solution of problem (1). Moreover, in order to compute approximately, the blow-up time $T_b$ of problem (1), we need only to estimate the blow-up time of problem (2).

3. The discrete problem

In fact, little attention has been devoted to the numerical study for problem (2), however, the numerical blow-up times of problem (2) has been studied by some authors, see for instance, [1, 11].

In order to compute the approximate values of blow-up solutions and blow-up times of problem (1), we can use finite difference operators to get the discrete problem of problem (1) as follows:

For $J$ a positive integer, we set $J = 1/h$ and we defined the grids

$$x_j = jh, \quad 0 \leq j \leq J$$

and $t_0 = 0, \quad t_{n+1} = t_n + k_n, \quad n = 0, 1, \ldots$ where $k_n > 0$, is the time steps, and we denote to the approximate value of $u$ and $w$ at the point $(x_j, t_n)$ by $U^n_j, W^n_j$ respectively.

We approximate the time derivative $w_t$ by the forward finite difference operator, while the second order derivatives by the standard second order centre finite difference operators. Thus, problem (2) can be written in discrete forms follows:

$$W^n_j = W^n_j + 2W^n_j + W^n_{j-1}$$

and equation (3) becomes

$$W^n_j = 0, \quad \forall n$$

$$W^n_j = 0, \quad \forall n$$

and

$$W^n_j = 0, \quad \forall n$$

$$W^n_j = 0, \quad \forall n$$

4. Blow-up in the discrete problem

The solution $W^n = (W^n_0, W^n_1, W^n_2, \ldots, W^n_J) \in R^{J+1}$, of the difference equation (4) does not exist for all $n \in N$, because there exists $m \in N$ such that $W^n$ become unbounded for $n \geq m$, see [1].

Remark 4.1

It is clear that the solution of problem (5), $U^n = (U^n_0, U^n_1, U^n_2, \ldots, U^n_J) \in R^{J+1}$, Can be computed only if $W^n$ is bounded.

Definition 4.2

Let $\{W^n\}_{n \geq 0}$ be a nonnegative solution of (4), with the time steps $[k_n]_{n \geq 0}$. We say that $\{W^n\}_{n \geq 0}$ achieves blow-up in finite time, if there exist $m \in N$, such that

i- $\lim_{n \rightarrow m} ||W^n||_{\infty} = \infty,$

ii- $T^n_m = \sum_{n=0}^m (k_n) < \infty,$

where the time $T^n_m$ is called the numerical blow-up time, and

$$||W^n||_{\infty} = \max_{0 \leq j \leq J} |W^n_j|.$$ In fact, the numerical blow-up time depends on the spatial grid $h$ and also on the choice of time steps. In [1], under certain assumptions, it has been proved that numerical solution of problem (4) convergent to the classical solution of problem (2), moreover, it has been studied the convergence of the numerical blow-up time of the discrete problem to the theoretical blow-up time of problem (2), which means

$$\lim_{h \rightarrow 0} T^n_m = T_b, \text{ and } \lim_{h \rightarrow 0} W^n_j = w(x_j, t_n)$$

5. Numerical Scheme

The finite difference equation (4), can be rewritten in the explicit Euler formula as follows:

$$W^n_{j+1} = r_n W^n_j + (1 - 2r_n) W^n_j + r_n W^n_{j-1}$$

where $r_n = k_n/h^2$

$$W^n_0 = W^n_1 = 0, \quad \forall n$$

$$W^n_0 = u_0(x_j)$$

It is well known that, $r_n \leq 1/2$, is the stability condition of the explicit Euler method for the heat equation. In [2], it has pointed out that time stepping based on a fixed step can lead to a different behavior from theoretical blow-up properties. To overcome this problem, we will consider the time step procedure considered first in [1] as follows:

$$k_n = \min \left( \frac{h^2}{2 \alpha}, \frac{h^2}{||W^n||_{\infty}} \right)$$

where $\alpha$ is a fixed positive constant.

It is well known that, for each fixed time interval $[0, T]$, where the solution $u$ of (2) is defined and sufficiently smooth, the numerical schemes (explicit Euler method) considered approximate $u$ with a rate of convergence of $O(T + h^2)$, where $T = \max k_n$. Because of the choice of $h$, we have a rate of convergence $O(h^2)$, as $h \rightarrow 0$.

The same order of convergence might be expected for the numerical blow-up times.

For any $n \in N$, in order to compute the numerical blow-up solution of problem (1), and the numerical blow-up time, we can use the following algorithm steps:

1- Use explicit Euler formula (6), to compute $W^n$, and the numerical blow-up time for $w$, $T^n_m = \sum_{n=0}^m (k_n)$, which can be taken at the first time

where $||W^n||_{\infty} \geq 10^{15}$.

2- Unless $W^n$ is unbounded, substitute it in (5), we get a linear system and from solving this system we get

$$U^n = (U^n_0, U^n_1, U^n_2, \ldots, U^n_J)$$

$\forall n > 0$

3- The numerical blow-up time for $u, T^n_k = \sum_{n=0}^k (k_n)$ can be taken at the first time, where $||U^n||_{\infty} \geq 10^{15}$.

6. Numerical experiment

In this section, we present some numerical approximation to the blow-up solution and blow-up time of problem(1), with the initial function $u_0(x) =$
\[-\frac{20}{\pi^2} \sin \pi x, \text{ which implies } w_0(x) = u_0'(x) = 20 \sin \pi x.\]

So from the maximum principles we can show that, the solution of problem (1), \(u\), and the solution of problem (2), \(w\), are negative and positive functions in \((0,1)\) respectively, see \([8,9]\).

Moreover, It is clear that \(u_0\) takes its minimum value at the point \(x = 1/2\), while \(u_0'\) takes its maximum at this point. Therefore, according to the known blow-up results for the semilinear heat equation (see \([8]\)), the blow-up in problem (2), occurs only at a single point, which is \(x = 1/2\).

This problem will be solved numerically by using the algorithm, which was suggested in section 5, with using Matlab programming. We will consider different choices for the positive parameter \(\alpha\) in the time stepping procedure (7), in order to examine experimentally, if there exists any rate of convergence for the numerical blow-up times with respect to the mesh size \(h\).

In tables (1), (2) and (3), with respect to the corresponding to meshes 10, 20 and 40 subintervals respectively, every four rows of the table correspond to the use of indicated values for \(\alpha\) in the time stepping procedure (7). In the first column, we show numerical blow-up times of problem (2), which arise from using explicit Euler method (6), and in the second column, we refer to the last iteration before numerical blow-up occurs by \(m\). In the third column, we show numerical blow-up times of problem (1), which arise from solving the linear systems (5), and in the last column, we refer to the last iteration before numerical blow-up occurs by \(k\).

In table (4), The errors in the numerical blow-up times, of problem (1), are computed by using

\[E_j = |T_j^k - T_j^\alpha| \quad \ldots \ldots \ldots \quad (8)\]

where \(T_j^k\) refer to the numerical blow-up time, with respect to \(J = \frac{1}{h}\), while \(T_j^\alpha\) refer to the numerical blow-up time, with respect to \(2J\).

We will consider two values for \(h\), \(h = 10\), and \(h = 20\).

Table 1,

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<th>(m)</th>
<th>(T_k)</th>
<th>(k)</th>
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Table 2,

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<th>(m)</th>
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<th>(k)</th>
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Table 3,

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Table 4,

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The next figures show the evolutions of the numerical blow-up solutions \(u\) & \(w\), of problems (1) & (2) respectively, which arise from using Euler explicit method, for different values to \(J\) and \(\alpha\).

![Figure 1](image)
Conclusions
From the numerical results, we can point out the following conclusions

1- The numerical blow-up time of problem (1), is mostly larger than the corresponding numerical blow-up times of problem (2), and $k > m$.

2- Decreasing the values of $\alpha$, leads almost to decreasing the number of iterations, $(k$ and $m)$ until the numerical blow up occurs, and increasing the numerical blow-up times.

3- The table of error (4), in the computed blow-up times, that was computed using (8), shows that: for a fixed value of $J$, the latest error can be get where $\alpha = \frac{3}{2}$. On the other hand, for a fixed values for $\alpha$ and $J$, we have $E_{ij} < E_j$.

4- The figures (1-4) show that, the blow-up in problem (2) occurs only at a single point and that confirm the theoretical results see [4], also, we can see experimentally, that problem (1) exhibits fixed point blow-up.

References
Global Existence and Steady States", Birkhuser Advanced Texts, Birkhuser,
Basel. 2007.