Properties of a Subclass of Meromorphic Univalent Functions By Using Multiplier transformation

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Abstract

In this present paper, we introduced and defined properties of a subclass of meromorphic univalent Functions defined by multiplier transformation in the puncture unit disk $\Delta^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. We obtain some properties, like, theorem of coefficient inequality, linear combination, extreme points and convex set.

Key words: coefficient inequality, linear combination, extreme points and convex set.

خواص الاصناف الجزئية للدوال الميرومورفية احادية التكافوء باستخدام التحويلات المضاعفة

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الخلاصة

في البحث نحن قدمنا وعرفنا خواص الاصناف الجزئية للدوال الميرومورفية احادية التكافوء المعرفة بواسطة التحويلات المضاعفة في قرص الوحدة المثقوب

$\Delta^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$ حصلنا على بعض الخواص مثل متراجحة المعاملات التركيب الخطي النقاط الحرجة والمجموعة المحدبة

الكلمات المفتاحية : متراجحة المعاملات، التركيب الخطي، النقاط الحرجة ،والمجموعة المحدبة .

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Introduction

Let *MF* denote the class of meromorphic functions f of the form

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \ , \qquad (1)$$

defined on the punctured unit disk $\Delta^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

Also, denote by Ω the subclass of *MF* consisting of functions of the from

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \ , \quad (a_m \geq 0\,). \qquad (2)$$

Now, we define on Ω multiplier transformation, we define the operator $L_1(r,\gamma)$ by the following infinite series when

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

then

$$L_1(r,\gamma)h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{m+\gamma}{1+\gamma}\right)^r a_m z^m \ \ (\gamma \geq 0\,). \qquad (3)$$

The operator $L_1(r,\gamma)$ was considered by Cho and Srivastava [3] and Cho and Kim [2].

Definition (1): The function $k \in \Omega$ be of the form (2) is said to be in the new class $L_1(\tau,\alpha,\mu,r,\gamma)$ if it satisfies the following condition:

$$\left| \frac{\dfrac{\dfrac{z^2\tau}{2}(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))} - \tau}{\alpha - \dfrac{\dfrac{z^2\alpha\mu}{2}(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))}} \right| < 1, \qquad (4)$$

for $0 < \mu \leq \frac{1}{2}$, $0 < \tau < 1, 0 < \alpha < 1 and\ r \in z = \{\dots, -1, 0, 1, \dots\}$.

The following interesting geometric properties of this function subclass were studied by several authors for another classes, like, Darus [2], Atshan [1].

Now**,** we obtain the necessary and sufficient condition for a function h to be in the class $L_1(\tau,\alpha,\mu,r,\gamma)$.

Theorem (1): Let $h \in \Omega$. Then $h \in L_1(\tau,\alpha,\mu,r,\gamma)$ if and only if

$$\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)a_m \le \alpha(1-\mu) \qquad (5)$$

where $0<\mu\le\frac{1}{2}, 0<\tau<1, and\ 0<\alpha<1.$

The result is sharp for the fumctiom

$$h(z)=\frac{1}{z}+\frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}m(m-1)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)}z^m, m\in\mathbb{N}.$$

Proof: For $|z|=1$, we have

$$\left|\frac{z^2\tau}{2}\left(L_1(r,\gamma)(h)(z)\right)''-\tau\left(L_1(r,\gamma)(h)(z)\right)\right| - \left|\alpha\left(L_1(r,\gamma)(h)(z)\right)-\frac{z^2\alpha\mu}{2}\left(L_1(r,\gamma)(h)(z)\right)''\right|$$

$$=\left|\sum_{m=1}^{\infty}\left(\frac{\tau}{2}(m^2-m)-\tau\right)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$-\left|\alpha\left(z^{-1}+\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right)-\frac{z^2\alpha\mu}{2}\left(2z^{-3}+\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_n m(m-1)z^{m-2}\right)\right|$$

$$=\left|\sum_{m=1}^{\infty}\left(\frac{\tau}{2}m(n-1)-\tau\right)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$-\left|\alpha z^{-1}+\alpha\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m-\alpha\mu z^{-1}-\sum_{m=1}^{\infty}\frac{\alpha}{2}\mu m\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$=\left|\sum_{m=1}^{\infty}(\frac{\tau}{2}m(m-1)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$-\left|(\alpha-\alpha\mu)z^{-1}-\sum_{m=1}^{\infty}(-\alpha+\frac{\alpha\mu m}{2}(m-1))\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m\right|$$

$$\leq\sum_{m=1}^{\infty}(\frac{\tau}{2}m(m-1)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m-(\alpha-\alpha\gamma)+\sum_{m=1}^{\infty}(-\alpha+\frac{\alpha\mu m}{2}(m-1))\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m$$

$$=\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^{r}(\frac{\tau}{2}m(m-1)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1))a_m-\alpha(1-\mu)\leq 0.$$

by hypothesis. Hence, $h\in L_1(\tau,\alpha,\mu,r,\gamma)$.

Conversely, assume that $h\in L_1(\tau,\alpha,\mu,r,\gamma)$, then from (4), we have

$$\left|\frac{\frac{\frac{z^2\tau}{2}(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))}-\tau}{\alpha-\frac{\frac{z^2\alpha\mu}{2}(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))}}\right|$$

$$=\left|\frac{\sum_{m=1}^{\infty}\quad(\frac{\tau}{2}(m^2-m)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m}{(\alpha-\alpha\mu)z^{-1}-\sum_{m=1}^{\infty}\quad(-\alpha+\frac{\alpha\mu m}{2}(m-1))\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m}\right|<1.$$

Since Re (z) ≤ $|z|$ ∀ z (z ∈ Δ^*), we get

$$\text{Re}\left\{\frac{\sum_{m=1}^{\infty}\quad(\frac{\tau}{2}(m^2-m)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m}{(\alpha-\alpha\mu)z^{-1}-\sum_{m=1}^{\infty}\quad(-\alpha+\frac{\alpha\mu m}{2}(m-1))\left(\frac{m+\gamma}{1+\gamma}\right)^{r}a_m z^m}\right\}\leq 1. \quad (6)$$

We can choose value of z on the real axis $\frac{z^2(L_1(r,\gamma)(h)(z))''}{(L_1(r,\gamma)(h)(z))} \in$ Re.

$$\sum_{m=1}^{\infty}(\frac{\tau}{2}(m^2-m)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^r a_m z^m$$
$$\leq (\alpha-\alpha\mu)z^{-1}-\sum_{m=1}^{\infty}(-\alpha+\frac{\alpha\mu m}{2}(-1+m))\left(\frac{m+\gamma}{1+\gamma}\right)^r a_m z^m$$

Let Re z $\rightarrow 1^-$

$$\sum_{m=1}^{\infty}(\frac{\tau}{2}(m^2-m)-\tau)\left(\frac{m+\gamma}{1+\gamma}\right)^r a_m$$
$$\leq -\sum_{m=1}^{\infty}\left(\frac{n+\gamma}{1+\gamma}\right)^r(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(-1+m))a_m+\alpha(1-\mu).$$

we can write (6) as

$$\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^r(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(-1+m))a_m \leq \alpha(1-\mu).$$

Finally,

$$h_m(z) = z^{-1} + \frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^r(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(n-1))} z^n, m = 1,2,\ldots \quad (7)$$

Corollary (1): Let $h \in L_1(\tau,\alpha,\mu,r,\gamma)$ Then

$$a_m \leq \frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^r(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1))}, m = 1,2,\ldots . \quad (8)$$

In the following theorem, we will show the class $L_1(\tau,\alpha,\mu,r,\gamma)$ is linear combination

Theorem (2): Let

$$h_i(z) = z^{-1} + \sum_{m=1}^{\infty} a_{m,i}\, z^m \ \in L_1(\tau,\alpha,\mu,r,\gamma) \ \ i \in \{1,2,\dots,\ell\} \text{ and}$$

$0 < c_i < 1$,

such that

$$\sum_{i=1}^{\ell} c_i = 1.$$

Then

$$H = \sum_{i=1}^{\ell} c_i h_i(z)$$

is also in the class $L_1(\tau,\alpha,\mu,r,\gamma)$.

Proof: By Theorem (1) for every $i \in \{1,2,\dots,\ell\}$ we have

$$\sum_{m=1}^{\infty} \frac{\left(\frac{m+\gamma}{1+\gamma}\right)^r \left(\frac{\tau}{2}(m^2-m) - \tau - \alpha + \frac{\alpha\gamma m}{2}(-1+m)\right)}{\alpha(1-\mu)} a_{m,i} \le 1.$$

Since

$$H(z) = \sum_{i=1}^{\ell} c_i h_i(z) \quad = \sum_{i=1}^{\ell} c_i \left(z^{-1} + \sum_{m=1}^{\infty} a_{m,i}\ z^m \right)$$

$$= \frac{1}{z} + \sum_{m=1}^{\infty} \left(\sum_{i=1}^{\ell} c_i a_{m,i} \right) z^m .$$

Therefore

$$\sum_{m=1}^{\infty} \frac{\left(\frac{m+\gamma}{1+\gamma}\right)^r \left(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)}{\alpha(1-\mu)} \left(\sum_{i=1}^{\ell} c_i\, a_{m,i}\right)$$

$$= \sum_{i=1}^{\ell} c_i \left(\sum_{m=1}^{\infty} \frac{\left(\frac{n+\gamma}{1+\gamma}\right)^r \left(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma n}{2}(-1+m)\right)}{\alpha(1-\mu)} a_{m,i} \right)$$

$$\leq \sum_{i=1}^{\ell} c_i = 1.$$

Hence $H \in L_1(\tau,\alpha,\mu,r,\gamma)$ and the proof is complete.

In the following theorem, we obtain the extreme points of the class $L_1(\tau,\alpha,\mu,r,\gamma)$.

Theorem (3): Let $h_0(z) = \frac{1}{z}$ and

$$h_m(z) = z^{-1} + \frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^r \left(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(-1+m)\right)} z^m, (m \geq 1).$$

Then $h \in L_1(\tau,\alpha,\mu,r,\gamma)$, if and only if

$$h(z) = w_0\, h_0(z) + \sum_{m=1}^{\infty} w_n\, h_n(z), \quad \left(w_m \geq 0,\ w_0 + \sum_{m=1}^{\infty} w_m = 1\right).$$

Proof: Suppose that

$$h(z) = w_0 h_0(z) + \sum_{m=1}^{\infty} w_m\, h_m(z)$$

$$h(z)=\frac{1}{z}+\sum_{m=1}^{\infty}\frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(-1+m)\right)}z^{m}$$

then

$$\sum_{m=1}^{\infty}\frac{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)}{\alpha(1-\mu)}$$

$$\times w_{n}\frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)}$$

$$=\sum_{m=1}^{\infty}w_{m}=1-w_{0}\leq 1.$$

So by Theorem (1), $h\in L_{1}(\tau,\alpha,\mu,r,\gamma)$. Conversly, we suppose

$h\in L_{1}(\tau,\alpha,\mu,r,\gamma)$. By (8), we have

$$a_{\mathrm{m}}\leq\frac{\alpha(1-\mu)}{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)},m\geq 1.$$

Setting

$$w_{m}=\frac{\left(\frac{m+\gamma}{1+\gamma}\right)^{r}\left(\frac{\tau}{2}(m^{2}-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(n-1)\right)}{\alpha(1-\mu)}a_{m},\quad m\geq 1,$$

and

$$w_{0}=1-\sum_{m=1}^{\infty}w_{m}.$$

Then

$$h(z) = w_0 h_0(z) + \sum_{m=1}^{\infty} w_m \, h_m \, (z)$$

Then

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

$$h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\alpha(1-\mu)w_n}{\left(\frac{m+\gamma}{1+\gamma}\right)^r \left(\frac{\tau}{2}(m^2-m) - \tau - \alpha + \frac{\alpha\gamma m}{2}(-1+m)\right)} z^m$$

$$= \frac{1}{z} + \sum_{m=1}^{\infty} \quad (h_m - z^{-1}) w_m$$

$$= \frac{1}{z}\left(1 - \sum_{m=1}^{\infty} w_m\right) + \sum_{m=1}^{\infty} \quad w_m h_m$$

$$= z^{-1} w_0 + \sum_{m=1}^{\infty} \quad w_m h_m$$

$$h(z) = w_0 h_0(z) + \sum_{m=1}^{\infty} w_m \, h_m \, (z).$$

In the following theorem, we will prove the class $L_1(\tau, \alpha, \mu, r, \gamma)$, is a convex set.

Theorem (4): The class $L_1(\tau, \alpha, \mu, r, \gamma)$ is convex set.

Proof: Let f_1 and f_2 be the arbitrary elements of the class $L_1(\tau, \alpha, \mu, r, \gamma)$. Then for every k $(0 < k < 1)$, we will show that

$(1 - Q)h_1 + Qh_2 \in L_1(\tau, \alpha, \mu, r, \gamma)$.

Thus, we have

$$(1-Q)h_1+Qh_2=\frac{1}{z}-\sum_{m=1}^{\infty}[(1-Q)a_m+Qb_m]z^m.$$

Hence,

$$\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^r\left(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)\ [(1-Q)a_m+Qb_m]$$

$$=(1-Q)\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^r\left(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)a_m$$

$$+Q\sum_{m=1}^{\infty}\left(\frac{m+\gamma}{1+\gamma}\right)^r\left(\frac{\tau}{2}(m^2-m)-\tau-\alpha+\frac{\alpha\gamma m}{2}(m-1)\right)b_m$$

$$\leq(1-Q)\alpha(1-\mu)+\alpha(1-\mu)Q=\alpha(1-\mu).$$

References

1. W. G. Atshan,Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative II,J. Surrveys in Mathematics and its Aplictions, 3(2008), 67-77.
2. N. E. Cho and T. H. Kim, Multiplier transformations and strongly close-to-convex functions, Bulletin of the Korean Mathematical Society, 40 (3) (2003), 399-410.
3. N. E. Cho and H.M. Srivastava, Argument estimates of certain analytic functions defined by a class of multipier transformations, Math. Comput. Modelling, 37(1-2)(2003),39-49.
4. S. Najafzadeh and A. Ebadian, Convex family of meromorphically multivalent functions on connected sets, Math. Com. Mod., 57(2013), 301-305.