He's Variational iteration Method to Approximate Time Fractional Wave non Linear Like Equation with Variable Coefficient

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Abstract:

In this paper, we will consider $He's$ variation alliteration method (VIM) for solving time fractional non-linear wave like with variable coefficient. In this method we use an approximate value for time fractional derivative when find Lagrange multiplier. Three examples show the efficiency and the importance of the method.

Keywords: Variational iteration method, time fractional on linear wave like with variable coefficient, Caputo derivative

1- Introduction:

We introduce a basic idea underlying the variational iteration method for solving nonlinear differential equations. Consider the general equation:

$$Lu(x,y,z,t)+Nu(x,y,z,t)= g(x,y,z,t), \quad (1)$$

Where L is a linear differential operator, N is a nonlinear operator, and g is a given analytical function. The essence of the method is to construct a correction functional of the from

$$u_{n+1}(x,y,z,t)=u_n+\int_0^t \lambda(\xi,t)[Lu_n(x,y,z,\xi)+ N\tilde{u}_n(x,y,z,\xi)-g(x,y,z,\xi)]d\xi, \quad (2)$$

Where λ is aLagrange multiplier which can be identifiedptimally via the variational theory Inokuti and Sekine [1], u_n is the approximate solution and $\tilde{u}_n$ denotes the restricted variation, i.e. $\delta\tilde{u}_n=0$. After determining the Lagrange multiplier λ and selecting an appropriate initial function u_0 , the successive approximations u_n of the solution u can be readily obtained.

In this paper we give definition fractional derivative introduced by Caputos [5].

Definition 1.1. Fractional integral operator of order $\beta\geq 0$ is defined as

$$I_x^{\beta} f(x)=\frac{1}{\Gamma(\beta)}\int_0^x (x-\tau)^{\beta-1} f(\tau)d\tau \, \beta>0, \quad (3)$$

$\Gamma$is a gamma function.

Definition 1.2. Fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_x^{\beta} f(x)=\frac{1}{\Gamma(m-\beta)}\int_0^x (x-\tau)^{m-\beta-1}\frac{d^m f(\tau)}{d\tau^m}d\tau,$$
$$m-1<\beta\leq m, m\in\mathbb{N}, x>0 \quad (4)$$

α is the order of the derivative. For the Caputo's derivative we have:

$1-D^{\beta}C=0,$ $C$is constant,
$2-D^{\beta}x^{\alpha}=0, \ \alpha\leq\beta-1,$
$3-D^{\beta}x^{\alpha}=\dfrac{\Gamma(1+\alpha)}{\Gamma(1-\beta+\alpha)}x^{\alpha-\beta}, \quad \alpha>\beta-1,$

In this paper, we consider the following time-fractional nonlinear wave like equation:

$$\frac{\partial^{\beta}u}{\partial t^{\beta}}=\sum_{i,j}^{n} F_{1ij}\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(u_{xi},u_{xj})+$$
$$\sum_i^n G_{1i}(\mathrm{X},t,u)\frac{\partial^p}{\partial x_i^p}G_{2i}(u_{xi})+H(\mathrm{X},t,u)+s(\mathrm{X},t), \quad (5)$$

With the initial conditions

$$u(\mathrm{X},0)=a_0(\mathrm{X}), \quad u_t(\mathrm{X},0)=a_1(\mathrm{X}).$$

Here $\mathrm{X}=(x_1,x_2,\dots,x_n)$, F_{1ij} and G_{1i} are nonlinear function of X,t,u , F_{2ij} and G_{2i} are nonlinear ------- function of derivatives of x_i and x_j, while$H$ and $S$are nonlinear function,where β is paramters describing the order of the fractional time derivatives. In $1<\beta\leq 2$, equation Equation(5) reduce to the fractional nonlinear wave like equation. In this paper, we apply $He's$ variational iteration method to find approximate solutions for time fractional wave like with variable coefficient.

2- $He's$ variational iterationmethod for solving time fractional wave like with variable coefficient.

To convey the basic ideal for variational iteration method to solve nonlinear time fractional wave like equation first, if we assume that $H(\mathrm{X},t,u) = Z(\mathrm{X},t,u) - hu(\mathrm{X})$, where h is constant. Equation (5) can be written in the form:

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} = \sum_{i,j}^{n} F_{1ij} \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{xi}, u_{xj}) + \sum_{i}^{n} G_{1i}(\mathrm{X},t,u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{xi}) + Z(\mathrm{X},t,u) - hu(\mathrm{X}) + s(\mathrm{X},t), \quad (6)$$

Using the standard variational iteration method, we construct the following correction functional as

$$u_{n+1}(\mathrm{X},t) = u_n(\mathrm{X},\mathrm{t}) + \int_0^t \lambda(\xi,t)[\frac{\partial^{\beta} u}{\partial \xi^{\beta}} - \sum_{i,j}^{n} F_{1ij} \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{xi}, u_{xj}) - \sum_{i}^{n} G_{1i}(\mathrm{X},\xi,u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{xi}) - Z(\mathrm{X},\xi,u) + hu(\mathrm{X}) - s(\mathrm{X},\xi)]d\xi, \quad (7)$$

$$u_{n+1}(\mathrm{X},t) = u_n(\mathrm{X},\mathrm{t}) + \int_0^t \lambda(\xi,t)[\frac{\partial^{\beta} u}{\partial \xi^{\beta}} + hu(\mathrm{X}) - \sum_{i,j}^{n} F_{1ij} \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(\tilde{u}_{xi}, \tilde{u}_{xj}) - \sum_{i}^{n} G_{1i}(\mathrm{X},\xi,\tilde{u}) \frac{\partial^p}{\partial x_i^p} G_{2i}(\tilde{u}_{xi}) - Z(\mathrm{X},\xi,\tilde{u}) - s(\mathrm{X},\xi)]d\xi, \quad (8)$$

Now, we assume that

$$\frac{\partial^{\beta} u}{\partial \xi^{\beta}} \cong \Gamma(\beta) \frac{\partial^2 u}{\partial \xi^2}, \; 1 < \beta \le 2, \quad (9)$$

$If\ \beta = 2$, Equation (9) becomes:

$$\frac{\partial^2 u}{\partial \xi^2} = \Gamma(2) \frac{\partial^2 u}{\partial \xi^2}.$$

Substituting (9) in (8), we obtain

$$u_{n+1}(\mathrm{X},t) = u_n(\mathrm{X},\mathrm{t}) + \int_0^t \lambda(\xi,t)[\Gamma(\beta) \frac{\partial^2 u}{\partial \xi^2} + hu(\mathrm{X}) - \sum_{i,j}^{n} F_{1ij} \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(\tilde{u}_{xi}, \tilde{u}_{xj}) - \sum_{i}^{n} G_{1i}(\mathrm{X},\xi,\tilde{u}) \frac{\partial^p}{\partial x_i^p} G_{2i}(\tilde{u}_{xi}) - Z(\mathrm{X},\xi,\tilde{u}) - s(\mathrm{X},\xi)]d\xi,$$

$$\delta u_{n+1}(\mathrm{X},t) = \delta u_n(\mathrm{X},\mathrm{t}) + \delta \int_0^t \lambda(\xi,t)[\Gamma(\beta) \frac{\partial^2 u}{\partial \xi^2} + hu(\mathrm{X}) - \sum_{i,j}^{n} F_{1ij} \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(\tilde{u}_{xi}, \tilde{u}_{xj}) - \sum_{i}^{n} G_{1i}(\mathrm{X},\xi,\tilde{u}) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{xi}) - Z(\mathrm{X},\xi,\tilde{u}) - s(\mathrm{X},\xi)]d\xi,$$

$$\delta u_{n+1}(\mathrm{X},t) = \delta u_n(\mathrm{X},\mathrm{t}) - \Gamma(\beta) \frac{\partial \lambda(\xi,t)}{\partial \xi} \delta u_n(\mathrm{X},\xi) + \Gamma(\beta)\lambda(\xi,t) \frac{\partial \delta u_n(\mathrm{X},\xi)}{\partial \xi} + \Gamma(\beta) \int_0^t \frac{\partial^2 \lambda(\xi,t)}{\partial \xi^2} \delta u_n(\mathrm{X},\xi) d\xi + \int_0^t h\, \lambda(\xi,t) \delta u_n(\mathrm{X},\xi) d\xi, \quad (10)$$

Moreover, the stationary conditions are as follow

$$\Gamma(\beta) \frac{\partial^2 \lambda(\xi,t)}{\partial \xi^2}\bigg|_{\xi=t} + h\ \lambda(\xi,t)\bigg|_{\xi=t} = 0,$$

$$1 - \Gamma(\beta) \frac{\partial \lambda(\xi,t)}{\partial \xi}\bigg|_{\xi=t} = 0,$$

$$\Gamma(\beta)\lambda(\xi,t)|_{\xi=t} = 0,$$

Therefore, the general Lagrange multiplier can be readily identified by

$$\lambda(\xi,t) = \frac{\sqrt{h}}{\sqrt{\Gamma\beta}} \sin(\frac{\sqrt{h}(\xi-\mathrm{t})}{\sqrt{\Gamma\beta}}), \quad (11)$$

Substituting (11) in (7), we have the following iteration formula

$$u_{n+1}(\mathrm{X},t) = u_n(\mathrm{X},\mathrm{t}) + \int_0^t \frac{\sqrt{h}}{\sqrt{\Gamma\beta}} \sin\left(\frac{\sqrt{h}(\xi-\mathrm{t})}{\sqrt{\Gamma\beta}}\right)[\frac{\partial^{\beta} u}{\partial \xi^{\beta}} - \sum_{i,j}^{n} F_{1ij} \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{xi}, u_{xj})$$

$$-\sum_{i}^{n} G_{1i}\,(\mathrm{X},\xi,u)\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(u_{xi})$$

$$-Z(\mathrm{X},\xi,u)+hu(\mathrm{X})-s(\mathrm{X},\xi)]d\xi, \qquad (12)$$

And we get,

$$u_{n+1}(\mathrm{X},t)=u_n(\mathrm{X},\mathrm{t})+\int_0^t\frac{\sqrt{h}}{\sqrt{\Gamma\beta}}\sin(\frac{\sqrt{h}(\xi-\mathrm{t})}{\sqrt{\Gamma\beta}})\ [\frac{\partial^{\beta}u}{\partial\xi^{\beta}}$$

$$-\sum_{i,j}^{n}F_{1ij}\frac{\partial^{k+m}}{\partial x_i^{k}\partial x_j^{m}}F_{2ij}\,(u_{xi}\,,u_{xj})$$

$$-\sum_i^n G_{1i}\,(\mathrm{X},\xi,u)\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(u_{xi})-H(\mathrm{X},\xi,\mathrm{u})-s(\mathrm{X},\xi)]d\xi, \qquad (13)$$

Second, if we assume that

$$H(\mathrm{X},t,u)=Z(\mathrm{X},t,u)+hu(\mathrm{X}),$$

Where h is constant. Equation (5) can be written in the form:

$$\frac{\partial^{\beta}u}{\partial t^{\beta}}=\sum_{i,j}^{n}F_{1ij}\frac{\partial^{k+m}}{\partial x_i^{k}\partial x_j^{m}}F_{2ij}\,(u_{xi}\,,u_{xj})+\sum_{i}^{n}G_{1i}\,(\mathrm{X},t,u)\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(u_{xi})+Z(\mathrm{X},t,u)+hu(\mathrm{X})+s(\mathrm{X},t), \qquad (14)$$

Using the standard variational iteration method, we construct the following correction functional as:

$$u_{n+1}(\mathrm{X},t)=u_n(\mathrm{X},\mathrm{t})+\int_0^t\lambda(\xi,t)[\frac{\partial^{\beta}u}{\partial\xi^{\beta}}-\sum_{i,j}^{n}F_{1ij}\frac{\partial^{k+m}}{\partial x_i^{k}\partial x_j^{m}}F_{2ij}\,(u_{xi}\,,u_{xj})-\sum_{i}^{n}G_{1i}\,(\mathrm{X},\xi,u)\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(u_{xi})$$

$$-Z(\mathrm{X},\xi,u)-hu(\mathrm{X})-s(\mathrm{X},\xi)]d\xi, \qquad (15)$$

$$u_{n+1}(\mathrm{X},t)=u_n(\mathrm{X},\mathrm{t})+\int_0^t\lambda(\xi,t)[\frac{\partial^{\beta}u}{\partial\xi^{\beta}}$$

$$-hu(\mathrm{X})-\sum_{i,j}^{n}F_{1ij}\frac{\partial^{k+m}}{\partial x_i^{k}\partial x_j^{m}}F_{2ij}\,(\tilde{u}_{xi}\,,\tilde{u}_{xj})$$

$$-\sum_{i}^{n}G_{1i}\,(\mathrm{X},\xi,\tilde{u})\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(\tilde{u}_{xi})-Z(\mathrm{X},\xi,\tilde{u})-s(\mathrm{X},\xi)]d\xi, \qquad (16)$$

Now, we assume that

$$\frac{\partial^{\beta}u}{\partial\xi^{\beta}}\cong\Gamma(\beta)\frac{\partial^{2}u}{\partial\xi^{2}},\ 1<\beta\le 2, \qquad (17)$$

$if\ \beta=2$, Equation (15) becomes:

$$\frac{\partial^{2}u}{\partial\xi^{2}}=\Gamma(2)\frac{\partial^{2}u}{\partial\xi^{2}}.$$

Substituting (17) in (16), we obtain

$$u_{n+1}(\mathrm{X},t)=u_n(\mathrm{X},\mathrm{t})+\int_0^t\lambda(\xi,t)[\,\Gamma(\beta)\frac{\partial^{2}u}{\partial\xi^{2}}-hu(\mathrm{X})-\sum_{i,j}^{n}F_{1ij}\frac{\partial^{k+m}}{\partial x_i^{k}\partial x_j^{m}}F_{2ij}\,(\tilde{u}_{xi}\,,\tilde{u}_{xj})-\sum_{i}^{n}G_{1i}\,(\mathrm{X},\xi,\tilde{u})\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(\tilde{u}_{xi})-Z(\mathrm{X},\xi,\tilde{u})-hu(\mathrm{X})-s(\mathrm{X},\xi)]d\xi, \qquad (18)$$

$$\delta u_{n+1}(\mathrm{X},t)=\delta u_n(\mathrm{X},\mathrm{t})+\delta\int_0^t\lambda(\xi,t)[\,\Gamma(\beta)\frac{\partial^{2}u}{\partial\xi^{2}}-hu(\mathrm{X})-\sum_{i,j}^{n}F_{1ij}\frac{\partial^{k+m}}{\partial x_i^{k}\partial x_j^{m}}F_{2ij}\,(\tilde{u}_{xi}\,,\tilde{u}_{xj})$$

$$-\sum_{i}^{n}G_{1i}\,(\mathrm{X},\xi,\tilde{u})\frac{\partial^{p}}{\partial x_i^{p}}G_{2i}(u_{xi})$$

$$-Z(\mathrm{X},\xi,\tilde{u})-s(\mathrm{X},\xi)]d\xi,$$

$$\delta u_{n+1}(\mathrm{X},t)=\delta u_n(\mathrm{X},t)-\Gamma(\beta)\frac{\partial\lambda(\xi,t)}{\partial\xi}\delta u_n(\mathrm{X},\xi)+\Gamma(\beta)\lambda(\xi,t)\frac{\partial\delta u_n(\mathrm{X},\xi)}{\partial\xi}+\Gamma(\beta)\int_0^t\frac{\partial^2\lambda(\xi,t)}{\partial\xi^2}\delta u_n(\mathrm{X},\xi)d\xi-\int_0^t h\,\lambda(\xi,t)\delta u_n\,(\mathrm{X},\xi)d\xi, \qquad (19)$$

Moreover, the stationary conditions are as follow

$$\Gamma(\beta)\frac{\partial^2\lambda(\xi,t)}{\partial\xi^2}\bigg|_{\xi=t} - h\ \lambda(\xi,t)\bigg|_{\xi=t} = 0,$$

$$1-\Gamma(\beta)\frac{\partial\lambda(\xi,t)}{\partial\xi}\bigg|_{\xi=t} = 0,$$

$$\Gamma(\beta)\lambda(\xi,t)|_{\xi=t} = 0,$$

Therefore, the general Lagrange multiplier can be readily identified by

$$\lambda(\xi,t) = \frac{\sqrt{h}}{\sqrt{\Gamma\beta}}\sinh(\frac{\sqrt{h}(\xi-\mathrm{t})}{\sqrt{\Gamma\beta}}), \qquad (20)$$

Substituting (20) for correction functional (15), we have the following iteration formula:

$$u_{n+1}(\mathrm{X},t) = u_n(\mathrm{X},\mathrm{t}) + \int_0^t \frac{\sqrt{h}}{\sqrt{\Gamma\beta}}\sinh\left(\frac{\sqrt{h}(\xi-\mathrm{t})}{\sqrt{\Gamma\beta}}\right)[\frac{\partial^\beta u}{\partial\xi^\beta} - \sum_{i,j}^{n} F_{1ij}\frac{\partial^{k+m}}{\partial x_i^k\partial x_j^m}F_{2ij}(u_{xi},u_{xj}) - \sum_i^n G_{1i}(\mathrm{X},\xi,u)\frac{\partial^p}{\partial x_i^p}G_{2i}(u_{xi}) - Z(\mathrm{X},\xi,u) - hu(\mathrm{X}) - s(\mathrm{X},\xi)]d\xi, \qquad (21)$$

And we get

$$u_{n+1}(\mathrm{X},t) = u_n(\mathrm{X},\mathrm{t}) + \int_0^t \frac{\sqrt{h}}{\sqrt{\Gamma\beta}}\sin(\frac{\sqrt{h}(\xi-\mathrm{t})}{\sqrt{\Gamma\beta}})\ [\frac{\partial^\beta u}{\partial\xi^\beta} - \sum_{i,j}^{n} F_{1ij}\frac{\partial^{k+m}}{\partial x_i^k\partial x_j^m}F_{2ij}(u_{xi},u_{xj}) - \sum_i^n G_{-1i}(\mathrm{X},\xi,u)\frac{\partial^p}{\partial x_i^p}G_{2i}(u_{xi}) - H(\mathrm{X},\xi,\mathrm{u}) - s(\mathrm{X},\xi)]d\xi \qquad (22)$$

By the variational iteration formula(13),(22) and initial approximation, we will get that first iterative step is the exact solution, when $\beta = 2$, as shows in this paper.

3- Application and Results

***Example* 3.1.** Consider the one-dimensional time fractional nonlinear wave-like equation:

$$\frac{\partial^\beta u}{\partial t^\beta} = u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3) - 18u^5 + u$$

$$0 < x < 1,\ \ t > 0,\ \ 1 < \beta \le 2 \qquad (23)$$

With the initial condition

$$u(x,0) = e^x,\quad \frac{\partial u(x,0)}{\partial t} = e^x,$$

The exact solution when ($\beta = 2$)

$$u(x,t) = e^{x+t}$$

We make the correction functional and the stationary conditions for Equation (23), the Lagrange multiplier can be determined as

$\lambda(\xi,t) = \frac{1}{\sqrt{\Gamma\beta}}\sinh(\frac{\xi-\mathrm{t}}{\sqrt{\Gamma\beta}})$, where $h = 1$

$$u_1(x,y,t) = u_0(x,\mathrm{y},\mathrm{t}) + \int_0^t \frac{1}{\sqrt{\Gamma\beta}}\sinh\left(\frac{\xi-\mathrm{t}}{\sqrt{\Gamma\beta}}\right)[\frac{\partial^\beta u}{\partial\xi^\beta} - u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) - u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3) + 18u^5 - u]\,d\xi,$$

$$u_1(x,y,t) = e^x(1+t) + \int_0^{\mathrm{t}} \frac{1}{\sqrt{\Gamma\beta}}\sin h\left(\frac{\xi-\mathrm{t}}{\sqrt{\Gamma\beta}}\right)[e^x + \xi e^x]d\xi,$$

$$u_1(x,y,t) = e^x + te^x + \int_0^t [\frac{1}{\sqrt{\Gamma\beta}}\sinh\left(\frac{\xi-\mathrm{t}}{\sqrt{\Gamma\beta}}\right)e^x + \xi\frac{1}{\sqrt{\Gamma\beta}}\sinh\left(\frac{\xi-\mathrm{t}}{\sqrt{\Gamma\beta}}\right)e^x]d\xi$$

$$u_1(x,y,t) = e^x + te^x - e^x + e^x\cosh\left(\frac{\mathrm{t}}{\sqrt{\Gamma\beta}}\right) - e^x\,t + e^x\sqrt{\Gamma\beta}\sinh\left(\frac{t}{\sqrt{\Gamma\beta}}\right),$$

$$u_1(x,y,t) = e^x\cosh\left(\frac{\mathrm{t}}{\sqrt{\Gamma\beta}}\right) + e^x\sqrt{\Gamma\beta}\sinh\left(\frac{t}{\sqrt{\Gamma\beta}}\right),$$

When $\beta = 2, u_1(x,t) = e^{x+t}$, is the exact solution.

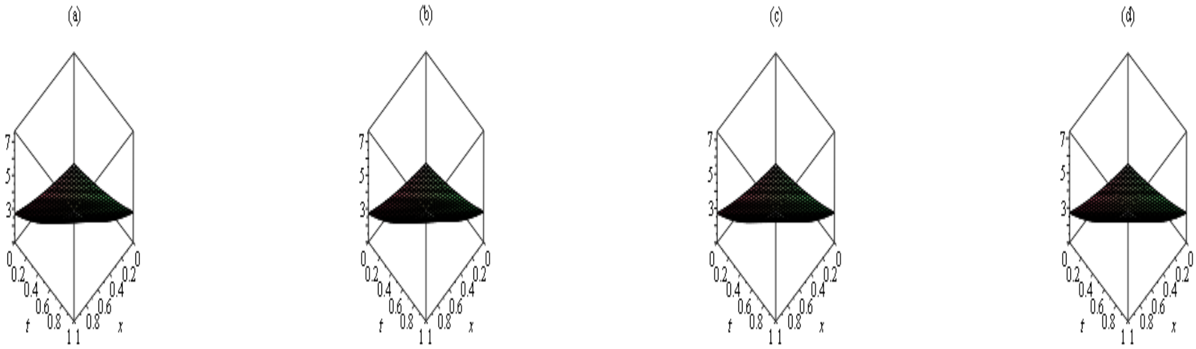

Fig.1.Approximate solutions of $u_1, (a)\beta = 1.3, (b)\beta = 1.5\ (c)\beta = 1.9\ (d)\beta = 2$.

x	$\beta = 1.2$	$\beta = 1.6$	$\beta = 1.8$	$\beta = 2$
0.1	3.080644524	3.106649985	3.067314574	3.004166024
0.2	3.404638737	3.433379216	3.389906864	3.320116923
0.3	3.762707719	3.794470861	3.746426482	3.669296670
0.4	4.158435145	4.193538846	4.140441595	4.055199968
0.5	4.595781587	4.634577175	4.575895638	4.481689072
0.6	5.079124154	5.121999910	5.057146781	4.953032424
0.7	5.613300303	5.660685343	5.589011550	5.473947392
0.8	6.203656250	6.256024817	6.176813026	6.049647464
0.9	6.856100474	6.913976692	6.826434125	6.685894443
1	7.577162855	7.641125967	7.544376467	7.389056099

Table.1. Approximate solutions ofu_1 with $t = 1$.

***Example* 3.2.** Consider the one-dimensional time fractional nonlinear wave-like equation:

$$\frac{\partial^\beta u}{\partial t^\beta} = x^2[\frac{\partial}{\partial x}(u_x u_{xx} - (u_{xx})^2] - u$$

$$0 < x < 1,\ \ t > 0,\ \ 1 < \beta \le 2 \qquad (24)$$

With the initial condition

$$\mathrm{u(x,0)} = 0, \quad \frac{\partial \mathrm{u(x,0)}}{\partial \mathrm{t}} = x^2 ,$$

The exact solution when ($\beta = 2$)

$$u(x,t) = x^2 \sin(t)$$

We make the correction functional and the stationary conditions for Equation(24), the Lagrange multiplier can be determined

$\lambda(\xi,t) = \frac{1}{\sqrt{\Gamma\beta}}\sin(\frac{\xi-\mathrm{t}}{\sqrt{\Gamma\beta}})$, where $h = 1$

$$u_1(x,y,t) = u_0(x,\mathrm{y},\mathrm{t}) + \int_0^t \frac{1}{\sqrt{\Gamma\beta}}\sin(\frac{\xi-\mathrm{t}}{\sqrt{\Gamma\beta}})\left[\frac{\partial^\beta u}{\partial t^\beta} - x^2[\frac{\partial}{\partial x}(u_x u_{xx} - (u_{xx})^2] + u\right] d\xi,$$

$$u_1(x,y,t) = x^2 t + x^2 \int_0^{\mathrm{t}} \frac{1}{\sqrt{\Gamma\beta}}\sin\left(\frac{\xi-\mathrm{t}}{\sqrt{\Gamma\beta}}\right)\xi\, d\xi,$$

$$u_1(x,y,t) = x^2\sqrt{\Gamma\beta}\sin\left(\frac{\mathrm{t}}{\sqrt{\Gamma\beta}}\right),$$

When $\beta = 2, u_1(x,t) = x^2\sin(t)$, is the exact solution

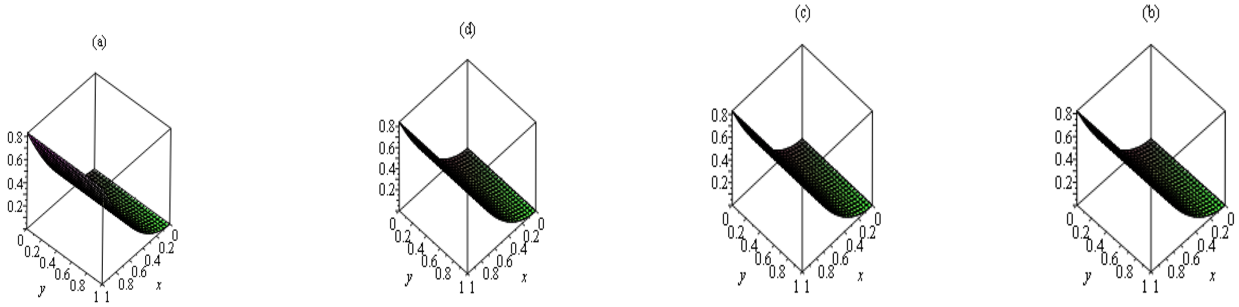

Fig.2.Approximate solutions of $\boldsymbol{u_1, with\ t = 1, (a)\beta = 1.2, (b)\beta = 1.5\ (c)\beta = 1.8\ (d)\beta = 2}$

x	$\beta = 1.3$	$\beta = 1.5$	$\beta = 1.7$	$\beta = 2$
0.1	0.008689614393	0.00866230835	0.008716113944	0.008912073601
0.2	0.034758457570	0.03464923340	0.034864455770	0.035648294400
0.3	0.078206529540	0.07796077514	0.078445025490	0.080208662410
0.4	0.139033830300	0.13859693360	0.139457823100	0.142593177600
0.5	0.217240359800	0.21655770880	0.217902848600	0.222801840000
0.6	0.312826118200	0.31184310060	0.313780102000	0.320834649600
0.7	0.425791105300	0.42445310920	0.427089583200	0.436691606400
0.8	0.556135321200	0.55438773430	0.557831292400	0.570372710500
0.9	0.703858765800	0.70164697630	0.706005229400	0.721877961700
1	0.868961439300	0.86623083500	0.871611394400	0.891207360100

Table. Approximate solutions of $\boldsymbol{u_1}$ **with** $\boldsymbol{t = 1.1}$.

***Example* 3.3.** Consider the two-dimensional time fractional nonlinear wave-like equation

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial^2}{\partial x\, \partial y}\left(u_{xx}u_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\, u_x u_y\right) - u$$

$0 < x < 1,\ \ t > 0,\ \ 1 < \beta \le 2$(25)

With the initial condition

$$u(x, y, 0) = e^{xy},\quad \frac{\partial u(x, y, 0)}{\partial t} = e^{xy},$$

The exact solution when ($\beta = 2$)

$$u(x, t) = e^{xy}(sint + cost),$$

We make the correction functional and the stationary conditions for equation (25), the Lagrange multiplier can be determined as

$\lambda(\xi, t) = \frac{1}{\sqrt{\Gamma\beta}}\sin(\frac{\xi - t}{\sqrt{\Gamma\beta}})$, where $h = 1$

$$u_1(x, y, t) = u_0(x, y, t) + \int_0^t \frac{1}{\sqrt{\Gamma\beta}}\sin(\frac{\xi - t}{\sqrt{\Gamma\beta}})\left[\frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^2}{\partial x\, \partial y}\left(u_{xx}u_{yy}\right) + \frac{\partial^2}{\partial x \partial y}\left(xy\, u_x u_y\right) + u\right] d\xi,$$

$$u_1(x, y, t) = e^{xy}(1 + t) + \int_0^t \frac{1}{\sqrt{\Gamma\beta}}\sin\left(\frac{\xi - t}{\sqrt{\Gamma\beta}}\right)[e^{xy} + \xi e^{xy}]d\xi,$$

$$u_1(x, y, t) = e^{xy} + te^{xy} + \int_0^t [\frac{1}{\sqrt{\Gamma\beta}}\sin\left(\frac{\xi - t}{\sqrt{\Gamma\beta}}\right)e^{xy} + \xi\frac{1}{\sqrt{\Gamma\beta}}\sin\left(\frac{\xi - t}{\sqrt{\Gamma\beta}}\right)e^{xy}]d\xi,$$

$$u_1(x, y, t) = e^{xy} + te^{xy} - e^{xy} + e^{xy}\cos\left(\frac{t}{\sqrt{\Gamma\beta}}\right) - e^{xy}\,t + e^{xy}\sqrt{\Gamma\beta}\sin\left(\frac{t}{\sqrt{\Gamma\beta}}\right),$$

$$u_1(x, y, t) = e^{xy}\cos\left(\frac{t}{\sqrt{\Gamma\beta}}\right) + e^{xy}\sqrt{\Gamma\beta}\sin\left(\frac{t}{\sqrt{\Gamma\beta}}\right),$$

When $\beta = 2, u_1(x, t) = e^{xy}(sint + cost)$, is the exact solution.

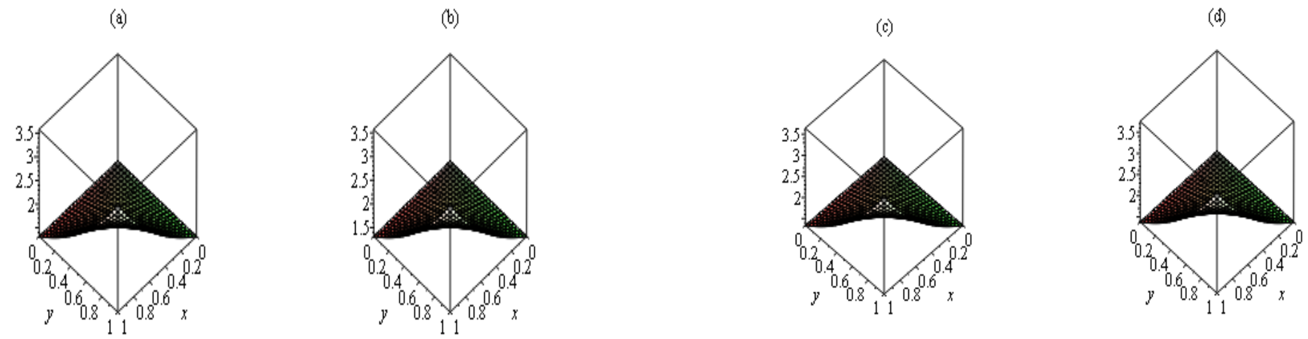

Fig.3.Approximate solutions of u_1, $with\ t = 1, (a)\beta = 1.3, (b)\beta = 1.6\ (c)\beta = 1.8\ (d)\beta = 2$

x	y	$\beta = 1.2$	$\beta = 1.5$	$\beta = 1.9$	$\beta = 2$
0.1	0.1	1.297068601	1.270425196	1.330867486	1.358318981
0.2	0.2	1.336570220	1.309115403	1.371398435	1.399685952
0.3	0.3	1.405097642	1.376235186	1.441711537	1.471449386
0.4	0.4	1.506978716	1.476023495	1.546247418	1.578141504
0.5	0.5	1.648897357	1.615026952	1.691864162	1.726761851
0.6	0.6	1.840627961	1.802819170	1.888590861	1.927546387
0.7	0.7	2.096159364	2.053101640	2.150780877	2.195144535
0.8	0.8	2.435389727	2.385363788	2.498850870	2.550394089
0.9	0.9	2.886679259	2.827383271	2.961900061	3.022994487
1	1	3.490715732	3.419012081	3.581676455	3.655554867

Table.3 . Approximate solutions of u_1 with $t = 1.1$.

4-Conclusion:

In this work, the variational iteration method is used to solve time fractional nonlinear wave like equation with variable coeffcient. This method gives the solution the first step i-e $u_1(x, t)$is the exact solution in case $\beta = 2$. Results show the ability and efficiency of this method.

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الملخص : في هذا البحث سوف نطبق طريقة أسلوب التغاير التكراري لحل معادلة الحرارة الموجة الخاصة اللاخطية الكسرية الزمن، وفي هذه الطريقة وضع قيمة تقريبية للمشتقة الكسرية الزمنية عند استخراج مضروب لاكرانج . أمثلة تبرهن كفاءة وأهمية هذه الطريقة.

الكلمات المفتاحية: طريقة أسلوب التغاير التكراري، معادلة الحرارة الخاصة الكسرية اللاخطية ،مشتقة كباتو

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