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Hollow Modules With Respect to an Arbitrary Submodule

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Abstract

In this paper ,we introduce hollow modules with respect to an arbitrary submodule .Let M be a non-zero module and T be a submodule of M .We say that M is aT-hollow module if every proper submodule K of M such that $T \nsubseteq K$ is a T-small submodule of M .We investigate the basic properties of a T-hollow module .

Keywords: T-small submodule, T-maximal submodule, T-radical submodule.

المقاسات المجوفة بالنسبة الى مقاس جزئي افتراضي

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الخلاصة

في هذا البحث قدمنا مفهوم المقاسات المجوفة بالنسبة الى مقاس جزئي افتراضي, ليكن M مقاسا غير صفري و T مقاس جزئي من M , يقال ان M مقاس مجوف بالنسبة للمقاس الجزئ T اذا كان لكل مقاس جزئي فعلي K من M بحيث ان المقاس الجزئي T غير محتوى في المقاس الجزئي K يكون مقاسا جزئيا صغيرا بالنسبة الى المقاس الجزئي T . افترضنا الخصائص الاساسية للمقاسات المجوفة بالنسبة الى مقاس جزئي افتراضي T .

1-Introduction

Throughout this paper , rings are associative with identity and modules are unital left R-modules. Recall that a submodule N of an R-module M is small, denoted by $N \ll M$,if for any submodule X of M , N+X = M implies that X = M .More detailes about small submodules can be found in [1-3]. The concept of small submodule has been generalized by some researchers , for this see [4,5]. In [6], the authors introduced the concept of small submodule with respect to an arbitrary submodul .Recall that a submodule N of M is called T-small in M , denoted by $N \ll_T M$, in case for any submodule $X \leq M$, $T \subseteq N+X$ implies that $T \subseteq X$ .In this paper ,we develop the properties of T-maximal submodules and introduce the concept of T-hollow module .Recall that a submodule K of M is called T-maximal in M if (T+K) / K is simple R-module ,see[6].Recall that the intersection of all T-maximal submodule in M is denoted by $Rad_T M$,see[6].

In section 2,we develop the properties of T- maximal submodule and the T-radical submodule of a module M ,We show that if T is a finitely generated submodule of a module M and N be submodule of M such that $T \nsubseteq N$,then there is a T-maximal submodule of M containing N ,see Theorem 2.2.Also we prove that Ra is not T-small submodule of a module M ,where $a \in M$ if and only if there is a T-maximal submodule N in M such that $a \notin N$ and $T \subseteq Ra + N$,see Theorem 2.9.

In section 3, we study the class of T-hollow module .We prove that if N is a non-zero submodule

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of T-hollow module M such that T⊆N ,then N is T-hollow module ,see proposition 3.5.Also we prove that if T is finitely generated submodule of T-hollow module M ,then T is cyclic ,see proposition 3.10. We investigate the basic properties of T-hollow module .

Let R be a ring and M be a left R-module .If X⊆M ,then $X \leq M$, $X \lneq M$, $X \ll M$, $X \ll_T M$ and $Rad_T M$ denote X is a submodule of M , X is a proper submodule of M , X is a small submodule of M , X is a T-small submodule of M and the T-radical of M , respectively .

2-The T-Radical of a module

In this section ,we develop the basic properties of the T-maximal submodules and the T-Radical of a module M .Following [6] ,let T, K be submodules of a module M .K is T-maximal submodule of M if (T+K)/ K is simple .The intersection of all T-maximal submodules of M is denoted by $Rad_T M$.

Proposition 2.1. Let M be a module and A ,T be submodules of M such that A ⊈ T .Then A is T-maximal submodule of M if and only if A+T = A+ Rx ,for every x∈ A + T and x∉ A .

Proof .→) Suppose that A is T-maximal submodule of M .Let x∈A+T and x ∉ A .Then A ⊊ A+ Rx ⊆ A+T .But A+T/A is simple , then A+T = A+ Rx .

←) Suppose that A+T = A+ Rx ,for every x∈ A+T and x∉ A .To show that A+T / A is simple .Let N be submodule of A+T such that A ⊊ N .Then there is x∈ N and x∉ A .Therefore A ⊊ A+ Rx ⊆ N ⊆ A+T .Thus N = A+T .

Theorem 2.2. Let N and T be submodules of a module M such that T is finitely generated and T⊈N Then there is a T-maximal submodule of M containing N .

Proof. Let N and T be a submodules of M such that T is finitely generated and T ⊈ N .Consider the set **S** ={ K | K is a submodule of M such that T⊈K and N⊆K }. Since T⊈N ,then N∈ **S** .Thus **S**≠∅. Let $\{C_\alpha\}_{\alpha\in\Lambda}$ be a chain in **S** .To show that $\bigcup_{\alpha\in\Lambda}C_\alpha \in$ **S** .Clearly $\bigcup_{\alpha\in\Lambda}C_\alpha$ is submodule of M and N⊆ $\bigcup_{\alpha\in\Lambda}C_\alpha$ .To show that T⊈ $\bigcup_{\alpha\in\Lambda}C_\alpha$ .Assume that $T \subseteq \bigcup_{\alpha\in\Lambda}C_\alpha$ .Let $T = Rm_1+Rm_2+\ldots+Rm_n \subseteq \bigcup_{\alpha\in\Lambda}C_\alpha$, $m_1,m_2,\ldots,m_n \in T$ .Hence there exists $\alpha_1,\alpha_2,\ldots,\alpha_n \in\Lambda$ such that $m_1 \in C_{\alpha 1}$,$m_2 \in C_{\alpha 2}$,…,$m_n\in C_{\alpha n}$ .Let $C_{\alpha j} = \max\{C_{\alpha 1,}C_{\alpha 2},\ldots,C_{\alpha n}\}$.Thus $T\subseteq C_{\alpha j}$ which is a contradiction .Thus$\bigcup_{\alpha\in\Lambda}C_\alpha\in$**S** **.**By Zorn's Lemma **S** has maximal element , say H .

Claim that H is a T-maximal submodule of M .To show that .Since H∈**S** ,then T⊈H and hence H+T /H≠0 .Now let W ≤ H+T such that $H \lneq W \leq H+T$ .Then by maximality of H ,W ∉ S and hence T ⊆ W. So W = W + T and hence W = H + T .Thus H is a T-maximal submodue of M containing N .

Proposition 2.3. Let T , K be submodules of a module M .If K is a T-maximal submodule of M , then K is A-maximal submodule of M , for all submodule A of T such that A⊈K .

Proof . Let K be a T-maximal submodule of M and let A be a submodule of T such that A⊈K .Since A⊆T ,then (K+A / K)⊆(K+T/K) .But K+T / K is simple .Therefore either K + A / K = 0 and hence A ⊆ K which is a contradiction. or (K+A / K) = (K+T / K) .Since K is T-maximal ,then K+A / K is simple .Thus K is A-maximal submodule of M .

Proposition 2.4. Let A and B be submodules of a module M such that M = A + B .Then A is a B-maximal submodule of M if and only if A is a maximal submodule of M .

Proof . →) Let A be a B-maximal submodule of M .Then A+B/A is simple and hence M/A is simple. Thus A is a maximal submodule of M .

←) Let A be maximal submodule of M , then(M/A) = (A+B/A) is simple .Thus A is a B-maximal submodule of M .

Proposition 2.5. Let A and B be submodules of a module M such that M = A⊕B and B is simple .Then A is a B-maximal submodule of M .

Proof . Let M = A⊕B where B is simple .By the second isomorphism theorem ,(A+B /A)≅(B /A∩B) ≅B .But B is simple ,therefore A+B /A is simple .Thus A is a B-maximal submodule of M .

Proposition 2.6. Let T, K be submodules of a module M .Then T∩K is a T-maximal submodule of M if and only if K is a T-maximal submodule of M .

Proof.→)Suppose that T∩K is a T-maximal submodule of M .Then (T + (T∩K))/(T∩K) = T/T∩K is simple .By the second isomorphism theorem (T/T∩K)≅(T+K/K) .Hence T+K/K is simple .Thus K is a T-maximal submodule of M .

←) Suppose that K is T-maximal submodule of M.Then T+K/K is simple.By the second isomorphism theorem ,(T+K / K) ≅(T / T∩K) = (T+(T∩K) / T∩K) .Hence T+(T∩K) /(T∩K) is simple .Thus T∩K is a T-maximal submodule of M .

Proposition 2.7. Let T be a finitely generated submodule of a module M and a ∈M .Then T ⊆ Ra if and only if a belong to no T-maximal submodule of M .

Proof .→) Assume that there exists a T-maximal submodule K of M such that a ∈ K . Then Ra ⊆K .But T ⊆ Ra ,therefore T ⊆ Ra ⊆ K .Thus T+K/K = 0 which is a contradiction .

←) Suppose that a∈M and a belong to no T-maximal submodule of M .To show that T⊆Ra .Assume not .By Theorem 2.2, there exists a T-maximal submodule K of M such that Ra⊆K .Thus a∈K which is a contradiction .

Proposition 2.8. Let 0≠T be a finitely generated ideal of a ring R and let a be an idempotent element of R .Then either a or 1-a belong to a T-maximal ideal of R .

Proof . Assume not .Then for all T-maximal ideal M of R ,a ∉M and 1-a ∉M .By proposition 2.7,T ⊆ Ra and T ⊆ R(1-a) .Hence T ⊆ Ra ⋂ R(1-a) .But a is an idempotent element ,therefore Ra ⋂ R(1-a) = 0. So T = 0 which is a contradiction .

Theorem 2.9. Let T be a finitely generated submodule of a module M and a∈M .Then Ra is not T-small submodule of M if and only if there is a T-maximal submodule N of M such that a∉N and T ⊆ Ra + N.

Proof.→) Suppose that Ra is not T-small submodule of M ,then there exists K ≤ M such that T⊆ Ra +K and T⊈K .Now Let **F** ={ N | N ≤ M and T ⊆ Ra+N , T⊈N }.Clearly that K∈ **F** and hence **F**≠∅. Let $\{C_\alpha\}_{\alpha\in\Lambda}$ be a chain in **F** .To show that $\bigcup_{\alpha\in\Lambda}C_\alpha\in$ **F** .Clearly $\bigcup_{\alpha\in\Lambda}C_\alpha$ is submodule of M and T⊆ Ra + $\bigcup_{\alpha\in\Lambda}C_\alpha$ .To show that T $\not\subseteq\bigcup_{\alpha\in\Lambda}C_\alpha$. Assume that T $\subseteq\bigcup_{\alpha\in\Lambda}C_\alpha$. Since T is finitely generated ,then T = Rm_1+ Rm_2 +…+ Rm_n , $m_1,m_2,\dots,m_n\in$ T and hence there exists $\alpha_1,\alpha_2,\dots,\alpha_n\in\Lambda$ such that $m_1\in C_{\alpha1}$,$m_2\in C_{\alpha2}$,…,$m_n\in C_{\alpha n}$ .Let $C_{\alpha j}$ = max{$C_{\alpha1}$,$C_{\alpha2}$,…,$C_{\alpha n}$ }.Thus T ⊆ $C_{\alpha j}$ which is a contradiction . By Zorn's lemma **F** has a maximal element, say H .

Claim that H is a T-maximal submodule of M .Since H∈**F** , then T⊈H and hence H+T/H ≠ 0.Now, let W ≤ M such that H ⪇ W ≤ H+T .Since T ⊆ Ra+ H ⊆ Ra+ W and H is the maximal element of **F** ,then W∉**F** and hence T ⊆ W implies W⊆ H+T ⊆ W+T =W .So W=H+T .Thus H is a T-maximal submodule of M .Since T⊆ Ra+ H and T⊈H ,then Ra+ H≠H and hence a∉ H .

←) Suppose that there exists a T-maximal submodule N of M with a∉N and T⊆Ra+ N .Since T+N/N is simple ,then T⊈N .Thus Ra is not T-small submodule of M .

Before we give our next result ,we need the following theorem .

Theorem 2.10.[6] Let M be a right R-module and 0≠T be a proper finitely generated submodule of M .Then $\sum_{L\in A}L = \bigcap_{K\in B} K$,where A = {L ≤ M | L$\ll_T$M and L+K⊆T+K ,for all T-maximal submodule K of M } and B = {K ≤ M | K is an T-maximal submodule of M }.

Theorem 2.11. Let T be a submodule of a module M .Then $\sum_{L\in A}L = \sum_{L\in A_1}L$,where A={L≤M | L$\ll_T$M and L+K ⊆ T+K ,for all T-maximal submodule K of M} and A_1 = {L ≤ M | L$\ll_T$ M and either L ⊆ K or L+K = T+K ,for all T-maximal submodule K of M} .

Proof. Let L∈A .Then L$\ll_T$ M and L+K ⊆T+K for all T-maximal submodule K of M .Then L+K/ K ⊆T+K/ K .Since K is a T-maximal submodule of M ,then T+K/ K is simple and hence either L+K/ K = 0 implies that L⊆K or L+K/ K = T+K/ K implies that L+K = T+K .Therefore L∈A_1.

Now , let L∈A_1.Then L$\ll_T$ M .Let K be a T-maximal submodule of M .Then either L⊆K and hence L+K = K ⊆ T+K or L+K = T+K .Therefore L∈A .

Proposition 2.12. Let T be a finitely generated submodule of a module M and m∈M such that Rm +K⊆T+K ,for all T-maximal submodule K of M .Then Rm $\ll_T$ M if and only if m∈ Rad_TM .

Proof.→) Suppose that Rm$\ll_T$ M and Rm+K ⊆ T+K ,for all K is a T-maximal submodule of M . By Theorem 2.9.[6] , Rm⊆A and hence Rm⊆Rad_TM .

←) Let m∈Rad_TM .To show that Rm$\ll_T$M .Assume that Rm is not T-small submodule of M . By Theorem 2.9., then there exists a T-maximal submodule K of M with m∉ K which is a contradiction .Thus Rm is a T-small submodue of M .

Proposition 2.13. Let M be a module .Then Rad_TM $\ll_T$ M if and only if The sum of any family of submodules of M {C_α ; Where $C_\alpha\ll_T$M and C_α+K ⊆ T+K ,for all T-maximal submodule K of M } is T-small in M .

Proof.→) Assume that Rad_TM $\ll_T$M .Let $\{C_\alpha\}_{\alpha\in\Lambda}$ be a family of T-small submodules of M with C_α+ K ⊆ T + K ,for all T-maximal submodule K of M .Since $\sum_{\alpha\in\Lambda}C_\alpha\subseteq$ Rad_TM and Rad_TM $\ll_T$ M ,then $\sum_{\alpha\in\Lambda}C_\alpha\ll_T$ M .

←) Clear by Theorem 2.9.[6] .

3- The T-hollow module .

In this section , we develop the basic properties of the T-hollow module .

Definition 3.1.

Let M be a non-zero module and T be a submodule of M .We say that M is a T-hollow module if every submodule K of M such that $T \not\subseteq K$ is a T-small submodule of M .

Remarks 3.2. (a)

Let M be a non-zero module .Then M is M-hollow module if and only if M is hollow module

Proof. Clear .

It is known that Z as Z-module is not hollow module .Then Z is not Z-hollow module .

(b) A T-hollow module need not to be hollow module as the following example shows :

Consider the module Z_6 as Z-module .If T = $\{\bar{0},\bar{3}\}$,then one can easily show Z_6 is T-hollow module . But Z_6 is not hollow module .

Proposition 3.3. Let M be a module with submodules $K \le T \le L \le M$ .If $K \ll_T M$, then $K \ll_T L$.

Proof. Suppose that $K \ll_T M$ .To show that $K \ll_T L$ .Let $T \subseteq K+X$ for some $X \le L$ .Since $K \ll_T M$, then $T \subseteq X$ .Thus $K \ll_T L$.

Proposition 3.4. Let M be a T-hollow module and let N be a non-zero submodule of M such that $T \subseteq N$ .Then N is a T-hollow module .

Proof.Let M be a T-hollow module .To show that N is T-hollow module ,let L be a proper Submodule of N such that $T \not\subseteq L$ .Since M be a T-hollow module ,then $L \ll_T M$ .By proposition 3.4., then $L \ll_T N$ Thus N is T-hollow module .

Proposition 3.5 Let M be a T-hollow module and let $f:M \to M`$ be an epimorphism ,where M` is a non -zero module .Then M` is f(T)-hollow module .

Proof . Suppose that M is a T-hollow module and let $f:M \to M`$ be an epimorphism .To show that M` is f(T)-hollow .Let $N` \subsetneq M`$ such that $f(T) \not\subseteq N`$.To show that $N` \ll_{f(T)} M`$ .Let $f(T) \subseteq N` + X$,for some $X \le M`$.Then $f^{-1}(f(T)) \subseteq f^{-1}(N`+X)$ .Therefore $T+ Ker f \subseteq f^{-1}(N`) + f^{-1}(X)$ .Thus $T \subseteq f^{-1}(N`) + f^{-1}(X)$ .To show that $T \not\subseteq f^{-1}(N`)$.Assume $T \subseteq f^{-1}(N`)$ .Then $f(T) \subseteq N`$ which is a contradiction .Thus $T \not\subseteq f^{-1}(N`)$ .Since M is T-hollow module ,then $f^{-1}(N`) \ll_T M$ and hence $T \subseteq f^{-1}(X)$.Therefore $f(T) \subseteq X$ .Thus M` is f(T)-hollow module .

Proposition 3.6. Let T and K be submodules of a module M such that $K \subseteq T$. If K is T-small submodule of M and M/K is T/K-hollow module ,then M is T-hollow .

Proof . Assume that $K \ll_T M$ and M/K is T/K-hollow module .We want to show that M is T-hollow. Let $N \le M$ such that $T \not\subseteq N$ and let $T \subseteq N+X$ for some $X \le M$ .Then $T/K \subseteq (N+X)/K$ and hence $T/K \subseteq (N+K)/K + (X+K)/K$ .To show that $T/K \not\subseteq N+K/K$ .Assume that $T/K = N+K/K$ .Then $T = N+K$ and hence $T \subseteq N+K$ .Since $K \ll_T M$,then $T \subseteq N$ which is a contradiction .Thus $T/K \not\subseteq N+K/K$ .Since M/K is a T/K-hollow module, then $N+K/K \ll_{T/K} M/K$ .Therefore $T/K \subseteq X+K/K$ .Thus $T \subseteq X+K$ .Since $K \ll_T M$,then $T \subseteq X$ .Thus M is T-hollow module .

Before we give our next result ,we need the following proposition .

Proposition 3.7 [6]. Let M be an R-module with submodules $N \le K \le M$ and $T \le K$ .If $N \ll_T K$, then $N \ll_T M$.

Proposition 3.8. Let L and T be a submodules of a module M with $T \subseteq L$ .If L is T-hollow module , then either L is a T-small submodule of M or whenever $L/K \ll_{T/K} M/K$, for some submodule $K \le L$ with $T \neq K$,then L=K .But not both .

Proof. Suppose that there exists $K \subseteq L$ with $T \neq K$ and $L/K \ll_{T/K} M/K$ .To show that $L \ll_T M$ .Let $T \subseteq L+X$,for some $X \le M$ .Since L is T-hollow and $T \not\subseteq K$,then $K \ll_T L$ .By proposition 3.8.[6] ,then $K \ll_T M$ .By Theorem 2.16.[6] ,therefore $L \ll_T M$.

Now, let L be a T-hollow module and satisfies the condition 1 and 2.Assume $L \neq 0$.Since $L \ll_T M$,then $L/0 \ll_{T/0} M/0$ .By condition 2 ,L = 0 which is a contradiction .

Proposition 3.9. Let T be a finitely generated submodule of a module M .If M is a T-hollow module ,then T is cyclic.

Proof. Let $T = Rx_1+Rx_2+\ldots+Rx_n$, for $x_i \in M$,$\forall$ i=1,…,n .Then $T \subseteq Rx_1+Rx_2+\ldots+Rx_n$ .If $T \neq Rx_1$, then $T \not\subseteq Rx_1$ .Since M is a T-hollow module ,then $Rx_1 \ll_T M$ and hence $T \subseteq Rx_2+Rx_3+\ldots+Rx_n$ .Therefore $T = Rx_2+Rx_3+\ldots+Rx_n$ .So,we delete the component one by one until we have $T = Rx_i$, for some i .Thus T is cyclic .

Proposition 3.10. Let M be a module with unique T-maximal submodule H ,where T is a finitely generated submodule of M .Then M is T-hollow .

Proof.Suppose that M has aunique T-maximal submodule H,where T is afinitely generated submodule of M .To show that M is T-hollow module .Let L be a proper submodule of M such that $T \nsubseteq L$ .To show that $L \ll_T M$,let $T \subseteq L+K$, for some $K \leq M$ .If $T \nsubseteq K$,then there is a T-maximal containing K .By Theorem 2.2 .But H is the unique T-maximal in M , then $K \subseteq H$ .Since $T \nsubseteq L$,then by the same way $L \subseteq H$ and hence $L+K \subseteq H$ .Therefore $T \subseteq L+K \subseteq H$ and hence T+H/H is simple .This is a contradiction .So $T \subseteq K$ and $L \ll_T M$ .Thus M is T-hollow module .

Proposition 3.11. Let T be a non-zero submodule of a module M .If M is T-hollow module .Then T is indecomposable .

Proof. Suppose that there are proper submodules K and L of T such that $T = K \oplus L$ .Therefore $T \nsubseteq K$ .Since M is T-hollow module ,then $K \ll_T M$ .But $T \subseteq K \oplus L$,therefore $T \subseteq L$ and hence T = L .This is a contradiction .Thus T is indecomposable .

Proposition 3.12. Let N and T be submodules of a module M such that $N \subsetneq T$ .If M is T-hollow module andT/N is finitely generated ,then T is finitely generated .

Proof. Let $T/N = R(x_1+N)+R(x_2+N)+\ldots+R(x_n+N)$,$x_i \in T$,$\forall i=1,\ldots,n$ .Clearly $Rx_i \subseteq T$,$\forall i=1,\ldots,n$ and hence $Rx_1+ Rx_2+\ldots+ Rx_n \subseteq T$ .To show that $T \subseteq Rx_1+Rx_2+\ldots+ Rx_n$ .Let $t \in T$ .Since T/N is finitely generated ,then $t + N = r_1 (x_1+N) + r_2 (x_2+N) +\ldots+ r_n (x_n+N) = (r_1x_1 + r_2x_2 +\ldots+ r_nx_n) + N$ and hence $t = (r_1x_1 + r_2x_2 +\ldots+ r_nx_n) + n$,for some $n \in N$ .Therefore $T = (Rx_1 + Rx_2 +\ldots+ Rx_n) + N$ .Since $N \subsetneq T$,then $T \nsubseteq N$ .But M is T-hollow module ,so $N \ll_T M$ .Therefore $T \subseteq Rx_1 + Rx_2 +\ldots+ Rx_n$ .Thus $T = Rx_1 + Rx_2 +\ldots+ Rx_n$.

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