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Convergence order-unit of vector metric space

تقارب الترتيب الاحادي للفضاء المتري المتجة

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Abstract

In this paper , we introduce order- unit of vector metric space by using an order - unit and give some of their properties . Also we prove that the order unit continuous topology is order unit vectorial continuous .

المستخلص :-

في هذا البحث قدمنا الترتيب الاحادي للفضاء المتري المتجهة باستخدام الرتبه الاحادية واعطينا بعض خصائصها، كذلك برهنا بأن التبولوجي المستمر ذات الترتيب الاحادي يكون مستمر اتجاهيا برتبه احاديه .

1.Introduction

Metric space are very important in mathematics and applied sciences. In [1], a vector space is determined with a rang graph having values in a Riesz space . In [1] and [2] , some results in metric space theory are generalized to vector metric space theory , and the Baire Theorem and some fixed point theorems in vector metric space are given . Actually , the study of metric space having value on a vector space has started by Zabrejko in [5] . The distance map in the sense of Zabrejko takes values from an ordered vector space . We use the structure of lattice with the vector metric having values in Riesz spaces, then we have new results as mentioned above .

In this paper ,we introduce the definition of that we call it the order- unit - vector metric space and related results about convergent in order - unit - vector metric space. Also we prove that the order unit continuous topology is order unit vectorial continuous .

2.Definitions

We give in this section , basic definitions which will be used throughout the paper .

Definition 2.1 [3]

Let X be a non-empty set and E be a Riesz space . The function $\sigma: X \times X \to E$ is said to be a vector metric (or E-metric) if it is satisfying the following properties :-

1) $\sigma(a,b) = 0$ if and only $a = b$.
2) $\sigma(a,b) \leq (a,c) + (b,c)$ for all $a, b, c \in X$

Also the triple (X, σ, E) is said to be vector metric space .

For convergence properties of sequences in vector metric space , one could refer to the paper by Cevik et. [3] .

Definition 2.2[6]

If $\mathcal{F}$ is a real vector space , a cone in $\mathcal{F}$ is a nonempty subset $\mathcal{C} \subseteq \mathcal{F}$ with the following two properties :

1. $a\mathfrak{f} \in \mathcal{C}$ whenever $a \in [0, \infty)$ and $\mathfrak{f} \in \mathcal{C}$;
2. $\mathfrak{f} + w \in \mathcal{C}$ whenever , $w \in \mathcal{C}$.

An ordered vector space $(\mathcal{F}, \mathcal{F}^+)$ is pair consisting of a real vector space $\mathcal{F}$ and a cone $\mathcal{F}^+ \subseteq \mathcal{F}$ satisfying

3. $\mathcal{F}^+ \cap -\mathcal{F}^+ = \{0\}$.

Definition 2.3 [4]

Let $\mathcal{V}$ be an ordered real vector space . An element $0 < e \in \mathcal{V}$ is called an order - unit if for each $x \in \mathcal{V}$ there exists a $\lambda > 0$ such that $x \leq \lambda e$. The set of order units of $\mathcal{V}$ will be denoted by $ou(\mathcal{V})$.

3.The main result

We introduce the definition of that we call it the order- unit- vector metric space which will be used throughout the paper . We will write $u \gg 0$ if $u > o \in \mathcal{V}$, where $(\mathcal{X}, \eta, \mathcal{V})$ is an order- unit- vector metric .

Definition 3.1

Let $\mathcal{X}$ be a non-empty set and $\mathcal{V}$ be an ordered real vector space with $ou(\mathcal{V}) \neq \emptyset$. A function $\eta: \mathcal{X} \times \mathcal{X} \to \mathcal{V}$ is called an order- unit- vector metric or ($ou(\mathcal{V}$)-metric) if :-

1) $\eta(u, v) \gg 0$ for all $u, v \in \mathcal{X}$
2) $\eta(u, v) = 0$ if and only $u = v$ for all $u, v \in \mathcal{X}$.
3) $\eta(u, v) \ll (u, w) + (v, w)$ for all $u, v, w \in \mathcal{X}$

In this case , the triple $(\mathcal{X}, \eta, \mathcal{V})$ is called an order- unit- vector metric space .

Note 3.2

Let $\mathcal{V}$ be an ordered real vector space . If $\{v_n\}$ is a decreasing sequence in $\mathcal{V}$ such that $\inf v_n = v$, we will write $v_n \downarrow 0$.

Definition 3.4

A sequence $\{z_n\}$ in an order- unit- vector metric space $(\mathcal{X}, \eta, \mathcal{V})$ is said to be an ou -vectorial converges (or ou ($\mathcal{V}$-converges)) , where $u \gg 0$, if for some $u \in \mathcal{X}$, there is a sequence $\{v_n\}$ in $\mathcal{V}$ such that $v_n \downarrow 0$ and $\eta(z_n, z) \ll uv_n$ for all n .We will denote this ou ($\mathcal{V}$-converges) by $z_n \to z(\ll$ ou) .

Definition 3.5

A sequence $\{z_n\}$in $\mathcal{X}$ is called a ou($\mathcal{V}$-Cauchy) sequence , where $u \gg 0$, if there exists a sequence $\{v_n\}$ in $\mathcal{V}$ such that $v_n \downarrow 0$ and $\eta(z_n, z_{n+t}) \ll uv_n$ holds for all n and t .

Definition 3.5

An order- unit- vector metric space $(\mathcal{X}, \eta, \mathcal{V})$ is called ou($\mathcal{V}$-complete) , where $u \gg 0$, if each ou($\mathcal{V}$-Cauchy) sequence in $\mathcal{X}$ is ou ($\mathcal{V}$-convergence) .

Proposition 3.6

Let $(\mathcal{X}, \eta, \mathcal{V})$ be an order- unit- vector metric space and $\{z_n\}$ be sequence in $\mathcal{X}$, where $u \gg 0$, then $\{z_n\}$ converges to z if only if $\eta(z_n, z) \to 0$ ($\ll$ ou) as $n \to \infty$.

Proof

Suppose $\{z_n\}$ is convergent to z. Choose $z \in \mathcal{X}$ where $u \gg 0$. Then there exists $v_n \in \mathcal{V}$ such that $v_n \downarrow 0$ and $\eta(z_n, z) \ll uv_n$. This means $\eta(z_n, z) \to 0$ as $n \to \infty$ for all n .
Conversely , suppose that $\eta(z_n, z) \to 0$ as $n \to \infty$ for all n . For some $u \in \mathcal{X}$ there exists $v_n \in \mathcal{V}$ such that $v_n \downarrow 0$ and $\eta(z_n, z) \ll uv_n$ for all n, where $u \gg 0$. Hence $\{z_n\}$ converges to z . ∎

Proposition 3.7

Let the triple $(\mathcal{X}, \eta, \mathcal{V})$ be an order- unit- vector metric space .And $\{z_n\}$ be sequence in $\mathcal{X}$, where $u \gg 0$. Then limit of $\{z_n\}$ is unique if it exists.

Proof :-

Assume that sequence $\{z_n\}$ convergent to two points z and r . For some , $u \in \mathcal{X}$ then there exists $v_n \in \mathcal{V}$ such that $v_n \downarrow 0$ and $\eta(z_n, z) \ll \frac{uv_n}{2}$, $\eta(z_n, r) \ll \frac{uv_n}{2}$ for all n , where $u \gg 0$.

Now $\eta(z, r) \ll \eta(z_n, z) + \eta(z_n, r) \ll \frac{uv_n}{2} + \frac{uv_n}{2} \ll uv_n$

Hence $(z, r) \ll uv_n$. This mean $(z, r) = 0$, hence z = r ■

Proposition 3.8

Let $(\mathcal{X}, \eta, \mathcal{V})$ be an order- unit- vector metric space and $\{z_n\}$ be sequence in $\mathcal{X}$, where $u \gg 0$, if $\{z_n\}$ converges to z then $\{z_n\}$ is Cauchy sequence .

Proof :-

For some $u \in \mathcal{X}$ there is $v_n \in \mathcal{V}$ such that $v_n \downarrow 0$ and $(z_n, z) \ll \frac{uv_n}{2}$, where $u \gg 0$.

$$\eta(z_n, z_{n+t}) \ll \eta(z_n, z) + \eta(z_{n+t}, z) \ll \frac{uv_n}{2} + \frac{uv_n}{2} \ll uv_n$$

Hence $\{z_n\}$ is a Cauchy sequence . ■

Proposition 3.9

Let $(\mathcal{X}, \eta, \mathcal{V})$ be an order- unit- vector metric space and $\{z_n\}$ converge in , where $u \gg 0$, then every sub sequence of convergent sequence is convergent to the same limit .

Proof:

Let $\{z_n\}$ be convergent sequence in $\mathcal{X}$ and converges to the point $z \in \mathcal{X}$. Let $\{z_{n_d}\}$ be sub sequence of $\{z_n\}$. Since $\{z_n\}$ converges to z , so there is $v_{n_1} \in \mathcal{V}$ satisfying $v_{n_1} \downarrow 0$ for some $u_1 \in \mathcal{X}$ such that

$\eta(z_n, z) \ll u_1 v_{n_1}$ for all n_1, where $u_1 \gg 0$.

Let $\{z_{n_d}\}$ converges to r .

Then there is $v_{n_2} \in \mathcal{V}$ satisfying $v_{n_2} \downarrow 0$ for some $u_2 \in \mathcal{X}$ such that

$\eta(z_{n_d}, r) \ll u_2 v_{n_2}$ forall n_2 where $u_2 \gg 0$.

Let $uv_n = max\{u_1 v_{n1} , u_2 v_{n2}\}$

Now $(z, r) = \eta(z, z_{n_d}) + \eta(z_{n_d}, r) \ll u_1 v_{n_1} + u_2 v_{n_2} \ll uv_n$.

This implies $(z, r) \ll uv_n$. Hence $\{z_{n_d}\}$ converges to z . ■

Definition 3.10

Let $(\mathcal{X}, \eta, \mathcal{V})$ and $(\mathcal{Y}, \eta, \mathcal{U})$ be two an order- unit- vector metric spaces with $ou(\mathcal{V}) \neq \emptyset$. We say that the map $\varphi: \mathcal{X} \to \mathcal{Y}$ is an order - unit – vectorial continuous at z if $z_n \to z(\ll ou)$ in $\mathcal{X}$ implies $\varphi(z_n) \to \varphi(z)(\ll ou)$.

Proposition 3.11

Let $(\mathcal{X}, \eta, \mathcal{V})$ and $(\mathcal{Y}, \eta, \mathcal{U})$ be two an order- unit- vector metric spaces with $ou(\mathcal{V}) \neq \emptyset$. If the mapping $\varphi: \mathcal{X} \to \mathcal{Y}$ is order unit continuous topology , then φ is order unit vectorial continuous .

Proof

Suppose that $z_n \to z(\ll ou)$ in $\mathcal{X}$. Then there exists a sequence $\{v_n\}$ in $\mathcal{V}$ such that $v_n \downarrow 0$ *and* $\eta(z_n, z) \ll uv_n$ for all n , where $u \gg 0$. Let b any nonzero positive element in u. Since φ is order unit continuous topology , there exists *some* v in $\mathcal{V}$ such that $\eta(\varphi(x), \varphi(y)) \ll ub$ whenever $y \in \mathcal{X}$ and $\eta(x, y) \ll v$, where $u \gg 0$. Then there exist a sequence $\{v_n\}$ *in* $\mathcal{U}$ and $\varphi(z_n, z) \ll uv_n$ for all n, where $u \gg 0$. Hence , $\varphi(z_n) \to \varphi(z)(\ll ou)$. ■

Proposition 3.12

Let $(\mathcal{X},\eta,\mathcal{V}),(\mathcal{Y},\eta,\mathcal{U})$ *and* $(\mathcal{Z},\eta,\mathcal{D})$ be an order unit vectorial continuous , φ mapping of $\mathcal{X}$ into $\mathcal{Y}$ and ξ a mapping of $\mathcal{Y}$ into $\mathcal{Z}$. If φ is order unit vectorial continuous at x in $\mathcal{U}$ and ξ is order unit vectorial continuous at x , then $\xi \circ \varphi$ is order unit vectorial continuous at x .

Proof

Let us take $b \gg 0$ in $\mathcal{D}$. As ξ is order unit vectorial continuous at , there exist some a in $\mathcal{U}$ such that $\eta(\xi(\varphi(x),\xi(y)) \ll ub$ whenever $y \in \mathcal{Y}$ *and* $\eta(\varphi(x),y) \ll a,$ where $u \gg 0$.
As φ is order unit vectorial continuous at x , for every $a \gg 0$ in $\mathcal{U}$, there exist some c in $\mathcal{V}$ such that $\eta(\varphi(x),\varphi(d)) \ll ua$ whenever $d \in \mathcal{X}$ *and* $\eta(x,d) \ll c,$ where $u \gg 0$.
This implies that $(\xi(\varphi(x),\xi(\varphi(d)) \ll ub$. Therefore $\xi \circ \varphi$ is order unit vectorial continuous at x . ∎

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