MJPS, Vol. (3), No. (2), 2016

Direct Estimation for One-Sided Approximation By Polynomial Operators

Alaa A Auad[1], Mousa M. Khrajan[2]

[1]*University of al-Anbar, Faculty of education of pure science/ Department of mathematics*

[2]*University of Al-Muthana/ science college/ department of mathematics and applications computer*

[*]*Corresponding author*: *alaa.adnan66.aa@gmail.com)*

Received : 15/12/2015 ,Accepted: 12/8/2016 , Published:20/9/2016

Abstract

The we characterize some positive operators for one-sided approximation of unbounded functions in weighted space $L_{p,\alpha}(X)$.We give also , an estimation of the degree of best one-sided approximation in terms averaged modulus of continuity.

Keyword: positive operators, weight space, average modulus of continuity.

1.Introduction

Continuing our previous investigations on polynomial operators for one-sided approximation to unbounded functions in weighted space (see [5]), it is the aim of this paper to develop a notion of direct estimation polynomial approximation with constructs which fits, to gather with results (see [8] and [9]) for unbounded function approximation processes.

To this end, let X=[0,1], we denoted by $L_p(X)$, $(1 \leq p < \infty)$ be the space of all real valued Lebesgue functions $f: X \to \mathbb{R}$ such that:

$$\|f\|_p = \left(\int_X |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty \quad \ldots\ldots\ldots(1).$$

Now, let W be the suitable set of all weight functions on X, such that $|f(x)| \leq M\,\alpha(x)$, where M is positive real number and

$\alpha: X \to \mathbb{R}^+$ weight function, which are equipped with the following norm

$$\|f\|_{p,\alpha} = \left(\int_X \left|\frac{f(x)}{\alpha(x)}\right|^p dx\right)^{\frac{1}{p}} < \infty \quad \ldots\ldots\ldots(2).$$

We set

$$\Delta_h^k f(x) = \begin{cases} \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(x+mh) & if\ x, x+mh \in X \\ 0 & otherwise \end{cases} \quad \ldots\ldots(3)$$

the k[th] local modulus of continuity is denoted by

$$\omega_k(f,x,\delta)_{p,\alpha} = sup\left\{\left|\Delta_h^k f(t)\right|, t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\right\} \quad \ldots\ldots(4).$$

The k[th] averaged modulus is used in this paper :

$$\tau_k(f,\delta)_{p,\alpha} = \|\omega_k(f,\cdot,\delta)\|_{p,\alpha} \quad \ldots\ldots\ldots(5).$$

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{P}_n$ the set of all algebraic polynomials of degree less than or equal to $n \in \mathbb{N}$.

For an unbounded function $f \in L_{p,\alpha}(X)$ and $n \in \mathbb{N}$, the degree of best weighted approximation and the degree of best one-weighted approximation are defined respectively by :

$E_n(f)_{p,\alpha} = inf\{\|f - p_n\|_{p,\alpha}\ ; p_n \in \mathbb{P}_n\}$ ……………………………..(6)
$\tilde{E}_n(f)_{p,\alpha} = inf\{\|q_n - p_n\|_{p,\alpha}\ ; p_n, q_n \in \mathbb{P}_n and p_n(x) \le f(x) \le q_n(x)\}$………..(7).
It easy to verify that there are not linear operators for one-sided approximation in X. Some non-linear construction have been proposed in [3] and [6].
Let us consider the step function

$$\psi(x) = \begin{cases} 0 & if -1 < x \le 0 \\ 1 & if \quad 0 < x \le 1 \end{cases} \quad ……………………………...……(8)$$

fix two sequences of polynomials $\{p_n\}$ and $\{q_n\}$, $p_n, q_n \in \mathbb{P}_n$ such that
$p_n(x) \le \psi(x) \le q_n(x)\ , x \in [-1,1]$ …………………………………..……….(9)
and $C_n = \|f - p_n\|_{p,\alpha} \ \to 0\ , p = 1$ ……………………………...……..(10).
For the first one we work in space $L_{p,\alpha}(X)$. For $1 \le p < \infty$, we construct two different sequences of operators, for $x \in X$, $n \in \mathbb{N}$ and $f \in L_{p,\alpha}(X)$ define

$$g_n(f, x) = f(0) + \int_X\ p_n(t - x) f'_{+}(t) dt - \int_X\ q_n(t - x) f'_{-}(t) dt \quad …………....(11)$$

and

$$G_n(f, x) = f(0) + \int_X\ q_n(t - x) f'_{+}(t) dt - \int_X\ p_n(t - x) f'_{-}(t) dt \quad ………….(12)$$

it is clear $g_n(f), G_n(f) \in \mathbb{P}_n$, we will prove that
$g_n(f) \le f(x) \le G_n(f), x \in X$ and both
$\|f - g_n(f)\|_{p,\alpha} \le C_n \|f'\|_{p,\alpha}$ and $\|f - G_n(f)\|_{p,\alpha} \le C_n \|f'\|_{p,\alpha}$,where C_n is given in (10).
In the second case, for function $f \in L_{p,\alpha}(X)$, we construct operators :

$$P_t(f, x) = \int_X\ [f((1 - t)x + tu) - \omega(f, (1 - t)x + tu, t)] du \quad …...(13)$$

and

$$Q_t(f, x) = \int_X\ [f((1 - t)x + tu) + \omega(f, (1 - t)x + tu, t)] du \quad …...(14).$$

It is clear that $P_t(f, x), Q_t(f, x) \in \mathbb{P}_n$ and therefore we can define
$L_{n,t}(f, x) = g_n(P_t(f), x)$ ……………………………………...……(15)
and
$M_{n,t}(f, x) = G_n(Q_t(f), x)$ …………………………………….……(16),

Where g_n and G_n are given by (11) and (12) respectively. We will prove that
$L_{n,t}(f, x) \le f(x) \le M_{n,t}(f, x), x \in X$ and present the degree of best one-sided approximation of unbounded functions by operators $L_{n,t}(f, x)$ and $M_{n,t}(f, x)$, $x \in X$ in terms averaged modulus of continuity.

In the last years there has been interest in studying open problems related to one-sided approximations (see [1] , [2]).
We point out that other operators for one-sided approximations have constructed in [7].
In particular, the operators presented in [6] yield the non-optimal rate $O(\tau\left(f, \frac{1}{\sqrt{n}}\right))$ where is ones consider in [4] give the optimal rate, but without an explicit constant. The paper is organized as follows. In section (3) we calculate the degree of best one-sided approximation of unbounded functions by mean of the operators define (13) and (14). Finally in the some section, we consider the degree of the best one-sided approximation by mean of the operators defined in (15) and (16).

2.Auxiliary results
We shall the following auxiliary lemmas :
Lemma 2.1 : [3]
If $f \in \mathcal{R}[0,1]$, $t \in (0,1)$ and functions $P_t(f)$, $Q_t(f)$ are defined by (13) and (14) respectively, then $P_t(f) \le f(x) \le Q_t(f)$, $x \in [0,1]$ and

$$\max\{\|P_t'\|_p , \|Q_t'\|\} \le \frac{3}{\pi} \tau(f, t)_p \quad .$$

Lemma 2.2 : [3]
Let $\psi(x)$ be given in (8). For $x \in [-1,1]$ define $p_n(x) = T_n^-\ (arc\ cosx\ )$ and $q_n(x) = T_n^+\ (arc\ cosx\ )$. Then
$p_n, q_n \in \mathbb{P}_n$, $p_n(f) \le \psi(x) \le q_n(f)$, $x \in [-1,1]$ and
$\|q_n - p_n\|_{p,[-1,1]} \le \frac{4\pi^2}{n+2}$.
Let us formulate and prove the following basic lemmas, which we shall use to prove our main results.
Lemma 2.3 :
For $f \in L_{p,\alpha}(X), (1 \le \ p < \infty\ ), n \in \mathbb{N}$ and $n \ge 2$. Let $g_n(f)$ and $G_n(f)$ be as (11) and (12) respectively. Then $g_n(f), G_n(f)\ \in \mathbb{P}_n$ and
$g_n(f,x)\ \le f(x) \le G_n(f,x)$, $x \in X$.
Proof :
From (9), (10), (11) and (12), it is clear that $g_n(f), G_n(f)\ \in \mathbb{P}_n$.
Since
$g_n(f,x) = f(0) + \int_X\ p_n(t-x){f'}_+(t)dt - \int_X\ q_n(t-x){f'}_-(t)dt$
where $p_n, q_n\ \in \mathbb{P}_n$, such that $p_n(x) \le\ f(x) \le q_n(x)$, $x \in [-1,1]$ and
$\|p_n - q_n\|_p \to 0$
We have , $p_n(x) \le\ \psi(x) \le q_n(x)$, $x \in [-1,1]$,
thus

$$g_n(f,x) \le f(0) + \int_X\ \psi(t-x){f'}_+(t)dt - \int_X\ \psi(t-x){f'}_-(t)dt$$

$$= f(0) + \int_X\ \psi(t-x)f'(t)dt = f(0) + f(x) - f(0)$$

$$= f(x)\ .$$

Also,

$$f(x) = f(0) + f(x) - f(0) = f(0) + \int_X\ f'(t)dt$$

$$= f(0) + \int_X\ \psi(t-x)f'(t)dt$$

$$= f(0) + \int_X\ \psi(t-x){f'}_+(t)dt - \int_X\ \psi(t-x){f'}_-(t)dt$$

$$\le f(0) + \int_X\ p_n(t-x){f'}_+(t)dt - \int_X\ q_n(t-x){f'}_-(t)dt$$

$$= G_n(f,x)\ .$$

Lemma 2.4 :
For $f \in L_{p,\alpha}(X), (1 \le\ p < \infty\ ), n \in \mathbb{N}$ and $n \ge 2$. Let $g_n(f)$ and $G_n(f)$ be as (11) and (12) respectively. Then

$$max\{\|f - g_n(f)\|_{p,\alpha}\ , \|f - G_n(f)\|_{p,\alpha}\ \} \le\ C_n\|f'\|_{p,\alpha}\ .$$

Proof :
We have

$$|f(x) - g_n(f,x)| \le \int_{-x}^{1-x}(q_n(y) - p_n(y))|f'(x+y)|dy\ ,$$

putting $\xi_n(y) = q_n(y) - p_n(y)$ and by using Holder's inequality

$$(\|f - g_n(f)\|_{p,\alpha})^p \le \int_X\ \left|\frac{\int_{-x}^{1-x}\xi_n(y)|f'(x+y)|dy}{\alpha(x)}\right|^p dx$$

$$\le\ \int_X\ (\left|\int_{-x}^{1-x}\xi_n(y)dy\right|^{p-1})\ (\left|\frac{\int_{-x}^{1-x}\xi_n(y)|f'(x+y)|^p dy}{\alpha(x)}\right|)dx$$

$$\le (\int_{-1}^{1}|\xi_n(w)|^{p-1}\,dw)\ (\int_X\ \left|\frac{f'(z)}{\alpha(z)}\right|^p\ (\int_{z-1}^{z}\frac{\xi_n(y)}{\alpha(y)}dy)dz)$$

$$\leq \left(\int_{-1}^{1}|\xi_n(w)|^p\, dw\right)\left(\int_X \left|\frac{f'(z)}{\alpha(z)}\right|^p dz\right)$$

Thus

$$\|f-g_n(f)\|_{p,\alpha} \leq \left(\int_{-1}^{1}|\xi_n(w)|^p\, dw\right)^{\frac{1}{p}}\left(\int_X \left|\frac{f'(z)}{\alpha(z)}\right|^p dz\right)^{\frac{1}{p}},$$

hence

$$\|f-g_n(f)\|_{p,\alpha} \leq \|\xi_n\|_p \|f'\|_{p,\alpha} = C_n \|f'\|_{p,\alpha}.$$

Similarly, we prove that , $\|f-G_n(f)\|_{p,\alpha} \leq C_n\|f'\|_{p,\alpha}$.

3. Main results :

Let us explicitly formulate direct theorem estimates of the degree of best approximation with constraints of unbounded functions by polynomial operators.

Theorem 3.1 :

For $f \in L_{p,\alpha}(X), (1 \leq p < \infty), n \in \mathbb{N}$ and $n \geq 2$. Let $P_t(f)$ and $Q_t(f)$ be as (13) and (14) respectively. Then

$$max\{\|f-P_t(f)\|_{p,\alpha}, \|f-Q_t(f)\|_{p,\alpha}\} \leq C_1(t,p)\tau(f,t)_{p,\alpha} \text{ and}$$

$$\tilde{E}_n(f)_{p,\alpha} \leq C_k(t,p)\tau(f,t)_{p,\alpha}.$$

Proof :

As usual, take q such that $\frac{1}{p}+\frac{1}{q}=1$, from (13), (14) and Holder's inequality, we obtain

$$\left(t\|f-P_t(f)\|_{p,\alpha}\right)^p = t^p \int_X \left|\frac{f(x)-P_t(f,x)}{\alpha(x)}\right|^p dx$$

$$\leq t^p \int_X \left|\frac{Q_t(f,x)-P_t(f,x)}{\alpha(x)}\right|^p dx$$

$$\leq 2^p t^p \int_X \int_0^t \left|\frac{\omega(f,(1-t)x+tu,t)}{\alpha((1-t)x)}\right|^p du\, dx .$$

Put $y=(1-t)x$ implies $dy=(1-t)dx$

$$\left(t\|f-P_t(f)\|_{p,\alpha}\right)^p \leq \frac{2^p t^p}{1-t}\int_0^t \int_u^{1-t+u} \left|\frac{\omega(f,y,t)}{\alpha(y)}\right|^p dy\, du$$

$$\leq \frac{2^p t^{\frac{p}{q}}}{1-t}\int_0^t \int_X \left|\frac{\omega(f,y,t)}{\alpha(y)}\right|^p dy\, du$$

$$\leq \frac{2^p t^{\frac{p}{q}+1}}{1-t}\int_X \left|\frac{\omega(f,y,t)}{\alpha(y)}\right|^p dy$$

thus

$$\|f-P_t(f)\|_{p,\alpha} \leq \frac{2}{(1-t)^{\frac{1}{p}}}\left(\int_X \left|\frac{\omega(f,y,t)}{\alpha(y)}\right|^p dy\right)^{\frac{1}{p}}$$

$$= \frac{2}{(1-t)^{\frac{1}{p}}}\|\omega(f,.,t)\|_{p,\alpha} = \frac{2}{(1-t)^{\frac{1}{p}}}\tau(f,t)_{p,\alpha}$$

since $\frac{2}{(1-t)^{\frac{1}{p}}}$ constant depending on t and p, then

$\|f-P_t(f)\|_{p,\alpha} \leq C_1(t,p)\tau(f,t)_{p,\alpha}$.

Similarly, we can prove $\|f-Q_t(f)\|_{p,\alpha} \leq C_1(t,p)\tau(f,t)_{p,\alpha}$.

We go to the following inequality :

$$\tilde{E}_n(f)_{p,\alpha} \leq \|Q_t(f)-P_t(f)\|_{p,\alpha} \leq \|f-Q_t(f)\|_{p,\alpha} + \|f-P_t(f)\|_{p,\alpha}$$

$$\leq C_k(t,p)\tau(f,t)_{p,\alpha} .$$

Theorem 3.2 :

For $f \in L_{p,\alpha}(X), (1 \leq p < \infty), n \in \mathbb{N}$. Let $L_{n,t}(f)$ and $M_{n,t}(f)$ be as (13) and (14) respectively. Then

$$L_{n,t}(f) \leq f(x) \leq M_{n,t}(f), \quad x \in X,$$

$$max\left\{\left\|f-L_{n,t}(f)\right\|_{p,\alpha},\left\|f-M_{n,t}(f)\right\|_{p,\alpha}\right\}\leq (C_1(t,p)+\frac{3C_n}{t})\tau(f,t)_{p,\alpha}$$

and

$$\tilde{E}_n(f)_{p,\alpha}\leq (C_k(t,p)+\frac{6C_n}{t})\tau(f,t)_{p,\alpha} .$$

Proof :

Let $P_t(f)$ and $Q_t(f)$ be as in (13) and (14) respectively. Also, from (15) and (16) , it is clear $L_{n,t}(f), M_{n,t}(f)\in \mathbb{P}_n$.

Moreover, from (15), (16), theorem 3.1, lemma 2.3, lemma 2.4 and lemma 2.1, we have

$$L_{n,t}(f,x)=g_n(P_t(f,x))\leq (P_t(f,x)\ \leq f(x)$$
$$\leq Q_t(f,x)\leq G_n\big(Q_t(f,x)\big)=\ M_{n,t}(f,x)\ ,\quad x\in X.$$

Also,

$$\left\|f-L_{n,t}(f)\right\|_{p,\alpha}\leq \|f-P_t(f)\|_{p,\alpha}+\left\|P_t(f)-L_{n,t}(f)\right\|_{p,\alpha}$$
$$\leq C_1(t,p)\tau(f,t)_{p,\alpha}+\|f-g_n(P_t(f))\|_{p,\alpha}$$
$$\leq C_1(t,p)\tau(f,t)_{p,\alpha}+C_n\left\|P_t^{'}(f)\right\|_{p,\alpha}$$
$$=C_1(t,p)\tau(f,t)_{p,\alpha}+C_n\left\|\frac{P_t^{'}(f,.)}{\alpha(.)}\right\|_p$$
$$\leq C_1(t,p)\tau(f,t)_{p,\alpha}+\frac{3C_n}{t}\tau\left(\frac{f}{\alpha},t\right)_p$$
$$=C_1(t,p)\tau(f,t)_{p,\alpha}+\frac{3C_n}{t}\tau(f,t)_{p,\alpha}$$
$$=(C_1(t,p)+\frac{3C_n}{t})\,\tau(f,t)_{p,\alpha} .$$

The estimate for $\left\|f-M_{n,t}(f)\right\|_{p,\alpha}$ follows analogously.

Thus

$$\tilde{E}_n(f)_{p,\alpha}\leq \left\|M_{n,t}(f)-L_{n,t}(f)\right\|_{p,\alpha}$$
$$\leq \left\|f-L_{n,t}(f)\right\|_{p,\alpha}+\left\|f-M_{n,t}(f)\right\|_{p,\alpha}$$
$$\leq 2(C_1(t,p)+\frac{3C_n}{t})\,\tau(f,t)_{p,\alpha}$$
$$\leq (C_k(t,p)+\frac{6C_n}{t})\,\tau(f,t)_{p,\alpha} .$$

Theorem 3.3 :

For $f\in L_{p,\alpha}(X), (1\leq\ p<\infty), n\in\mathbb{N}, n\geq 2$. Let p_n and q_n be the sequence of polynomials constructed as in (9), set

$\mathcal{Z}_n(f)=L_{n,\frac{1}{n}}(f)$ and $\mathcal{H}_n(f)=M_{n,\frac{1}{n}}(f)$, where

$L_{n,\frac{1}{n}}(f)$ and $M_{n,\frac{1}{n}}(f)$ are given in (15) and (16) respectively. Then

$$\mathcal{Z}_n(f,x)\leq f(x)\leq\ \mathcal{H}_n(f,x)\ ,\quad x\in X,$$

$max\{\|f-\mathcal{Z}_n(f)\|_{p,\alpha},\|f-\mathcal{H}_n(f)\|_{p,\alpha}\}\leq (C_1(t,p)+\frac{3C_n}{t})\tau(f,\frac{1}{n})_{p,\alpha}$ and

$$\tilde{E}_n(f)_{p,\alpha}\leq 2(C_k(t,p)+\frac{12n\pi^2}{n+2})\tau(f,\frac{1}{n})_{p,\alpha} .$$

Proof :

From (15) and (16) with $t=\frac{1}{n}$ and $n\geq 2$, we obtain

$L_{n,\frac{1}{n}}(f,x)=g_n(P_{\frac{1}{n}}(f,x))$ and $M_{n,\frac{1}{n}}(f,x)=G_n(Q_{\frac{1}{n}}(f,x))$ where

$P_{\frac{1}{n}}(f), Q_{\frac{1}{n}}(f)\in\mathbb{P}_n$. So $g_n(P_{\frac{1}{n}}(f)), G_n(Q_{\frac{1}{n}}(f))\in\mathbb{P}_n$

From lemma 2.3, we have $g_n(f,x)\leq f(x)\leq G_n(f,x)\ ,\quad x\in X.$

Hence , $\mathcal{Z}_n(f,x)\leq f(x)\leq\mathcal{H}_n(f,x)\ ,\quad x\in X.$

We need an estimate for $\|f - \mathcal{Z}_n(f)\|_{p,\alpha}$ one has :
From (15), lemma 2.2 and theorem 3.2

$$\|f - \mathcal{Z}_n(f)\|_{p,\alpha} = \left\|f - L_{n,\frac{1}{n}}(f)\right\|_{p,\alpha} \leq (C_k(t,p) + \frac{3C_n}{\frac{1}{n}})\, \tau\left(f, \frac{1}{n}\right)_{p,\alpha} .$$

$$\leq (C_k(t,p) + \frac{12n\pi^2}{n+2})\, \tau\left(f, \frac{1}{n}\right)_{p,\alpha} .$$

Similarly, we can prove

$$\|f - \mathcal{H}_n(f)\|_{p,\alpha} \leq (C_k(t,p) + \frac{12n\pi^2}{n+2})\, \tau\left(f, \frac{1}{n}\right)_{p,\alpha} .$$

Thus

$$\tilde{E}_n(f)_{p,\alpha} \leq \|\mathcal{H}_n(f) - \mathcal{Z}_n(f)\|_{p,\alpha}$$
$$\leq \|\mathcal{H}_n(f) - f\|_{p,\alpha} + \|f - \mathcal{Z}_n(f)\|_{p,\alpha}$$
$$\leq 2(C_k(t,p) + \frac{12n\pi^2}{n+2})\, \tau\left(f, \frac{1}{n}\right)_{p,\alpha} .$$

References

[1]Bustamante, J. and Quesada, J.M., (2012), “Polynomial operators for one-sided approximation to functions in $W_p^r[0,1]$”, J. Math. Analy. and Appl., 393, pp. 517-525.

[2]Bustamante, J. and Quesada, J.M., (2012),”Best one-sided L_1 approximation to the Heaviside and functions”, J. Appro. Theory, 164(6), pp. 791-802.

[3]Bustamante, J., Quesada, J.M. and Reinaldo, M.C., (2014), “Polynomial operators for one-sided L_p approximation to Riemann integrable functions”, J. inequ. And Appl., 494.

[4]Hristov, V.H. and Ivanov, K.G., (1988), “Operators for one-sided approximation of functions”, In: Constructive theory of functions 87, proc. Intern. Conference, Varna, Sofia, PP. 222-232.

[5]Jassim, K. S. and Alaa. A.A., (2014),”On best one-sided approximation by interpolation polynomials in spline $L_{p,w}(X)$”,J.Mat. Theory and Modl.,Vol.4,No.4, pp.95-102.

[6]Lenze, B., (1988),” Operators for one-sided approximation by algebraic polynomials”, J. Approx. Theory, 54,pp. 169-179.

[7]Motornyi, V.P. and Pas’ko, A.N., (2004),” On the best one-sided approximation of some classes of differentiable functions in L_1”, J. Approx. Theory,10(1-2), pp. 159-169.

[8] Motornyi,V.P.and Pas’ko,A.N., (2010),”On one-sided approximation of a step by algebraic polynomials in the mean”,J.Ukra. Math., 62(3), pp. 476-482.

[9]Taikov, L.V., (2000),”One-sided approximation of functions”, Math. Notes, 67(1),pp.108-111.