On p-duo Semimodules

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Abstract

The concept of p-duo semimodule is introduced as a generalization of duo semimodule, where a semimodule *M* is said to be a p-duo if every pure subsemimodule of *M* is fully invariant. Many results about this concept are given.

Keywords: p-duo semimodule, duo semimodule, weak duo semimodule, pure semimodule.

الخلاصة

مفهوم شبه الموديول من النوع p ثنائي قد استحدث كتعميم ل ثنائي شبه الموديول حيث ان شبه الموديول يقال له بانه p ثنائي اذا كان كل شبه موديول جزئي نقي من شبه الموديول, تام الثبات. العديد من النتائج حول هذا المفهوم قد حصلت.

الكلمات المفتاحية: شبه الموديول من النوع p ثنائي، شبه موديول ضعيف ، شبه موديول نقي.

1-Introduction

throughout all semirings are commutative have identity and all semimodules are untital. R is a semiring and Ma left R-semimodule. A subsemimodule N of a semimodule M is called fully invariant if $f(N) \subseteq N$, for every R-endomorphism f of M. It is clear that 0 and M are fully invariant subsemimodules of M. The R-semimodule M is called duo if every subsemimodule of M is fully invariant. The semiring R is a duo if it is duo as R-semimodule. It is clear that every semiring is a duo semiring. Also we introduced the concept of weak duo semimodules, where an R-semimodule M is called weak duo if every direct summand subsemimodule of M is fully invariant.

Also, the concept of purely duo(shortly p-duo) semimodule is introduced where an R-semimodule M is called a p-duo if each pure subsemimodule of M is fully invariant where a subsemimodule N of M is said to be pure if $IM \cap N = IN$ for every ideal I of R. Also, p-duo semimodule, and some conditions under which p-duo and weak duo are equivalent is studied.

2-Preliminaries

Some definitions that needed in this paper, will be introduced.

Definition 2.1:[Chaudhari & Bonde, 20105]

Let R be a semiring. a left R-semimodule is a commutative monoid$(M,+)$ with additive identity 0_M for which we have a function $R \times M \to M$, defined by $(r,x) \mapsto rx$ (scalar multiplication), which satisfies the following conditions for all elements r and s of R and all elements x and y of M:

(i) $(rs)x = r(sx)$
(ii) $r(x+y) = rx + ry$
(iii) $(r+s)x = rx + sy$
(iv) $0_R x = 0 = r0 \; for\ all\ r \in R\ and\ x \in M$

If $1_R x = x$ hold for each $x \in M$ then the semimodule M is called unitary.

Definition 2.2:[Chaudhari & Bonde, 20105]

A non-empty subset N of a left R-semimodule M is called subsemimodule of M if N is closed under addition and scalar multiplication, that is N is itself a semimodule over R, (denoted by $N \hookrightarrow M$).

Definition2.3:[Golan, 2013]

Let R be a semiring and $L \hookrightarrow M$(R-semimodule). Then L is said to be a direct summand of M if there exists R-subsemimodule K such that $M = L \oplus K$ and M is called a direct sum of $L$and K.

Definition2.4:[Abdulameer, 2017]

A left R-semimodule is said to be semisimpleif it's a direct sum of its simple subsemimodule.

Definition2.5:[Ebrahimi & Shajari, 2010]

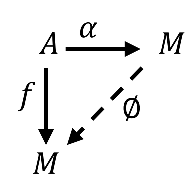
An R-semimodule M is called multiplication if for each subsemimodule N of Mthere exist some ideal $I$of R such that $IM = N$.

Definition2.6:[Katsov *et al.*, 2009]

If M is an R-semimodule then its left annihilator is $ann_R(M) = \{r \in R: rm = 0$ for every element $m \in M\}$.

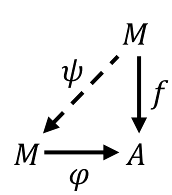
Definition2.7:[Abdulameer, 2017]

A semimodule M is called quasi-injective if for any R-semimodule A, any R-monomorphism $f: A \to M$ any R-homomorphism $\alpha: A \to M$, there exists R-homomorphism $\varphi: M \to M$ (endomorphism) such that $f = \alpha\varphi$.



Definition 2.8:[Althani, 2011]

A semimodule M is called quasi-projective if for any R-semimodule A, any R-epimorphism $\alpha: M \to A$ any R-homomorphism $\varphi: M \to A$, there exists R-homomorphism $\psi: M \to M$(endomorphism) such that $\alpha = \varphi\psi$.



Definition 2.9:[Abdulameer, 2017]

A subsemimodule N of M is said to be fully invariant if $f(N) \subseteq N$ for each R-endomorphism f on M.

Definition 2.10:[Abdulameer, 2017]

A semimodule M is said to be duo if each subsemimodule of M is fully invariant.

3- p-duo semimodules

In [Özcan & Harmanci, 2006; Anderson & Fuller, 1974] weak duo and p-duo modules were introduced respectively. Analogously, the similar concepts for semimodules is introduced.

Definition 3.1:

A semimodule M is called weak duo if every direct summand subsemimodule of M is fully invariant.

Definition 3.2:

A subsemimodule N of a semimodule M is called pure if $IM \cap N = IN$ for each ideal I of R.

Definition 3.3:

A semimodule M is called a p-duo if each pure subsemimodule of M is fully invariant.

Remark 3.4:

1-Every duo semimodule is p-duo and every p-duo is weakly duo.
2-Every multiplication semimodule is a duo semimodule, hence a p-duo semimodule and a weakly duo semimodule.
3-Every pure simple semimodule M is a p-duo semimodule, hence a weak duo semimodule.

Proposition 3.5:

A direct summand of p-duo semimodule is a p-duo.

Proof:

Let L be a direct summand of a p-duo R-semimodule. That is $M = L \oplus K$ for some $K \hookrightarrow M$. let N be a pure subsemimodule of L and let $f: L \to L$ be an R-homomorphism semimodule. Since L is a direct summand, then L is pure subsemimodule in M, hence N is a pure subsemimodule in M.
Defined $h = f\pi_L: M \to M$ by $h(x) = f(x)$
h is a well-defined R-homomorphism. It follows that $h(N) \subseteq N$, since M is a p-duo semimodule and N is a pure subsemimodule in M. But $h(N) = f(N), (N \hookrightarrow L)$. Hence $f(N) \subseteq N$; that is N is fully invariant subsemimodule of L. Thus L is a p-duo semimodule.

Lemma3.6:

If N is a fully invariant subsemimodule of M and if $M = K \oplus H$, then $N = (N \cap K) \oplus (N \cap H)$.

Proof:

Let $n \in N$, since $M = K \oplus H \Longrightarrow n = k + h$ and $\pi_K: M \to M \Longrightarrow \pi_K(N) \subseteq N$ (fully invariant)

$$\pi_K(n) = k \Longrightarrow k \in N \Longrightarrow k \in N \cap K$$

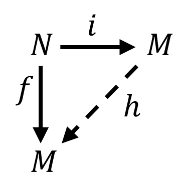
Similarly, $h \in N \cap H$

So $= (N \cap K) + (N \cap H)$ and $(N \cap K) \cap +(N \cap H) = N \cap (K \cap H) = N \cap (0) = 0$, so $N =$
$(N \cap K) \oplus (N \cap H)$. ◊

In [9] the purely quasi-injective of modules was introduced. Analogously, the similar concept for semimodules is introduced.

Definition 3.7:

An R-semimodule$M$ is called purely quasi-injective if every pure subsemimodule N of M and every $f: N \to M$, there exists an R-homomorphism $h: M \to M$ such that $h \circ i = f$ where i is the inclusion mapping.



Proposition 3.8:

Let M be an R-semimodule such that every cyclic subsemimodule is pure. Then M is a P-duo semimodule if and only if for each $f \in End(M)$ and for each $m \in M$, there exists $r \in R$ such that $f(m) = rm$.

Proof:

$\Longrightarrow$Let $f \in End(M), m \in M$. Since $< m >$ is pure (where $< m >$ denots the cyclic subsemimodule generated by m), then $f(< m >) \subseteq < m >$.Hence theresult is obtained.

$\Longleftarrow$ The stated condition implies $f(N) \subseteq N$ for every $f \in End(M)$. It follows that $M$is a duo semimodule. Hence it is a P-duo semimodule.
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Remark 3.9:

If M is a semisimple semimodule. Then the following statements are equivalent:

1-$M$is a duo semimodule.

2-$M$is a p-duo semimodule.

3-$M$is a weak duo semimodule.

Proposition 3.10:

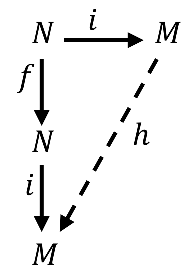
Let M be a P-duo R-semimodule. Then

1-If M is purely quasi-injective, then every pure subsemimodule of M is a P-duo semimodule.

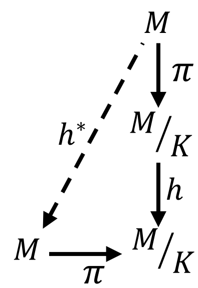
2-If M is quasi-projective, then for any pure subsemimodule $N$of M, $M/N$is a P-duo R-semimodule.

Proof:

1-Let N be a pure subsemimodule and K be a pure subsemimodule of N. Let $f: N \to N$ be a homomorphism. Since N is a pure subsemimodulein M and M is a purely quasi-injective semimodule, there exists $h: M \to M$such that $h \circ i = i \circ f$ where i is the inclusion mapping of N into M.



Thus $h \circ i(K) = h(K)$. But K is a pure subsemimodule in$N$ and N is a pure submodule in M, implies K is a pure subsemimodule in M. Hence $h(K) \subseteq K$.
Also $h \circ i(K) = i \circ f(K) = f(K)$. Thus $h(K) = f(K)$ and so $f(K) \subseteq K$. Therefore N is a P-duo semimodule.
2-LetL/Kbe a pure subsemimodule ofM/K. and let $h: M/K \to M/K$ be an R-homomorphism. Let$\pi : M \to M/K$be the natural epimorphism. Since M/K is quasi-



projective, there exists $h^*: M \to M$ such that $\pi \circ h^* = h \circ \pi$. Hence $h^*(m) + K = h(m + K)$ for each $m \in M$. But $L/K$is a pure subsemimodule in$M/K$and$K$ is a pure subsemimodule in M, so that L is a pure subsemimodule in M.

It follows that $h^*(L) \subseteq L$, since M is a P-duo semimodule. Hence $h\left(\frac{L}{K}\right) = h\big(\pi(L)\big) = \pi(h^*(L) = {h^*(L)}/{K} \subseteq L/K$, Thus L/K is a P-duo semimodule. ◊

Remark 3.11 :

Let a semimodule $M = L_1 \oplus L_2$be a direct sum of subsemimodules L_1, L_2. Then L_1 is fully invariant subsemimodule of M if and only if $Hom(L_1, L_2) = 0$

Proposition3.12:

Let a semimodule $M = L \oplus K$ be a direct sum of subsemimodules L, K such that M is a p-duo semimodule. Then $Hom(L, K) = 0$.

Proof:
Since L is a direct sum of M, L is a pure subsemimodule in M. But M is a p-duo semimodule, so L is fully invariant subsemimodule in M. Hence $Hom(L, K) = 0$ by note (3.11).

Lemma 3.13:
Let M be a semimodule. If $annM_1 + annM_2 = R$, with M_1, M_2 are two semimodule, then $N = I_1N \oplus I_2N$.

Proof:
Let $I_1 = annM_1, I_2 = annM_2$

$$I_1M \cap N = I_1N \subseteq I_1M = I_1(M_1 + M_2) = M_2$$

Similarly, $I_2M \cap N = I_2N \subseteq M_1$

$$\Longrightarrow I_1N \cap I_2N \subseteq M_1 \cap M_2 = (0)$$

Now, let $n \in N \Longrightarrow n = 1n = r_1n + r_2n. r_1 \in I_1, r_2 \in I_2 \;\Longrightarrow n \in I_1N + I_2N$
So $N = I_1N \oplus I_2N$.

Theorem3.14:
Let an R-semimodule $M = L_1 \oplus L_2$ be a direct sum of subsemimodules L_1, L_2 such that $annL_1 + annL_2 = R$. Then M is a p-duo semimodule if and only if L_1 and L_2 are p-duo semimodule and $Hom(L_i, L_j) = 0$ for $i \neq j, i, j \in \{1,2\}$.

Proof:
$\Longrightarrow$ By proposition(3.5) and Proposition(3.12).
$\Longleftarrow$ Let N be a pure subsemimodule of M. since $annL + annK = R$, then by lemma(3.13)$N = N_1 \oplus N_2$ for some $N_1 \hookrightarrow L, N_2 \hookrightarrow K$. Hence N_1 is a pure subsemimodule in L_1 and N_2 is a pure subsemimodule in L_2. Let $f: M \longrightarrow M$ be an R-homomorphism. Then $\rho_j f i_j: L_j \rightarrow L_j, j = 1,2$, where ρ_j is the canonical projection and i_j is the inclusion map. Hence $\rho_j f i_j(N_j) \subseteq N_j, j = 1,2$, since $L_j (j = 1,2)$ is a p-duo semimodule. Moreover by hypothesis $\rho_k\, f i_j(N_j)(N_2) = 0$ for $k \neq j (k, j \in \{1,2\})$. Then $f(N) = f(N_1) + f =$ $f(i_1(N_1)) + f(i_2(N_2)) = (\rho_1 + \rho_2)\left(f(i_1(N_1)) + f(i_2(N_2))\right) =$ $\rho_1\ (f(i_1(N_1)) + \rho_2\left(f(i_1(N_1))\right) + \rho_1\left(f(i_2(N_2))\right) + \rho_2\left(f(i_2(N_2))\right) =$ $\rho_1\ (f(i_1(N_1))) + \rho_2\left(f(i_2(N_2))\right) \subseteq N_1 + N_2 = N$. Thus M is a p-duo semimodule.
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Lemma 3.15:
Let M be an R-semimodule such that $M = \oplus_{i \in I} M_i$. If N is fully invariant subsemimodule of M, then $N = \oplus_{i \in I}(N \cap M_i)$.
Proof: As in Lemma (3.6).

Theorem 3.16:
Let a semimodule $M = \oplus_{i \in I} M_i$. Then M is a p-duo semimodule if and only if
1-$M_i$is a p-duo semimodule for all $i \in I$.
2-$Hom(M_i, M_j) = 0$ for all $i \neq j, j \in I$.
3- $N = \oplus_{i \in I}(N \cap M_i)$ for every pure subsemimodule $N$of M.

Proof:
$\Longrightarrow$ By proposition(3.5), proposition(3.12) and Lemma(3.15) .
$\Longleftarrow$let$N$ be a pure subsemimodule of M. By(3), $N = \oplus_{i\in I}(N \cap M_i)$. Thus $N \cap M_i$ is a pure subsemimodule in M_i. Let $f: M \longrightarrow M$. For any j$\in I$. Consider the following

$$M_j \xrightarrow{i_j} M \xrightarrow{f} M \xrightarrow{\rho_j} M_j$$

Where i_j is the inclusion map and ρ_j is the canonical projection. Hence $\rho_j f i_j: M_j \to M_j$ and so $\rho_j f i_j(N \cap M_i) \subseteq N \cap M_i$ for each $j \in I$. By (2), $Hom(M_i, M_j) = 0$ for all $i \neq j, j \in I$. Hence $f(\oplus_{j\in I}(N \cap M_j)) \subseteq \oplus_{j\in I}(\rho_j f i_j(N \cap M_i) = N$. Thus M is a p-duo semimodule. ◊

In [Al-Bahraany, 2000] the pure intersection property ofmoduleswas introduced. Analogously, the similar concept for semimodules is introduced.

Definition 3.17:
An R-semimodule M is said to satisfy pure intersection property (shortly PIP) if the intersection of any two pure subsemimodule is pure too.

Corollary 3.18:
Let $M = \oplus_{i\in I} M_i$. Then M is a p-duo semimodule if the following conditions hold:
1-$\oplus_{i\in \acute{I}} M_i$is a p-duo for every finite subset $\acute{I}$ of I.
2-Msatisfies PIP.

Proof:
By (1), M_i is a p-duo semimodule for every $i \in I$. Also $M_i \oplus M_j$ is a p-duo semimodule for each $i \neq j, i, j \in I$. Let $x \in N$, hence $x \in \oplus_{i\in \acute{I}} M_i = L$, for some finite subset $\acute{I}$of I. Thus $x \in N \cap L$. By (2), $N \cap L$ is a pure subsemimodule in M. But $N \cap L \subseteq L$, so $N \cap L$ is a pure subsemimodule in L. Since L is a p-duo semimodule by (1), $N \cap L$ is a fully invariant subsemimodule in L. Thus $N \cap L = \oplus_{i\in \acute{I}}[(N \cap L) \cap M_i] = \oplus_{i\in \acute{I}}(N \cap M_i)$. It follows that $x \in \oplus_{i\in \acute{I}}(N \cap M_i)$ and so $x \in \oplus_{i\in I}(N \cap M_i)$. Thus $N = \oplus_{i\in \acute{I}}(N \cap M_i)$ and hence M is a p-duo semimodule by Theorem(3.16).◊

In [Saad *et al.*, 1990] the summand sum property and summand intersection property of modules were introduced respectively. Analogously, the similar concepts for semimodules is introduced.

Definition 3.19:
An R-semimodule is said to satisfy summand sum property if $K + L$ is a direct summand of M whenever K and L are direct summands of M.

Definition 3.20:
An R-semimodule is said to satisfy summand intersection property if $K \cap L$ is a direct summand of M whenever K and L are direct summands of M.

Proposition 3.21:

Let M be a P-duo semimodule. If L is a direct summand of M and $N$is a pure subsemimodule of M, then $L \cap N$ is a pure subsemimodule of M.

Proof:

Since$L$ is a direct summand of $M, M = L \oplus H$ for some $H \hookrightarrow M$. Since M is a P-duo semimodule and K is a pure subsemimodule,by (Lemma) then K is a fully invariant. Hence $K = (K \cap L) \oplus (K \cap H)$. Thus $K \cap L$ is a direct summand of K, so $K \cap L$ is a puresubsemimodule in K. But K is a pure subsemimodule in M, hence $K \cap L$ is a pure subsemimodule in M.

Proposition 3.22:

Let M be an R-semimodule, then the following two statements are equivalent:

1-$M$is a p-duo semimodule.

2-For each two pure subsemimodule of M with zero intersection, then their sum is fully invariant in M.

Proof:

(1→2) It is clear.

(2→1) Let N be a pure subsemimodule of M. Let $H = (0)$, then H is a pure subsemimodule in M and $N \cap H = (0)$. Hence by (2), $N = N + H$ is a fully invariant. Thus M is a p-duo semimodule. ◊

Lemma 3.23:

An R-semimodule M satisfies PIPif $I(N \cap L) = IN \cap IL$, for each ideal I of R and for each pure subsemimodules N, L of M.

Proof:

Let N, L be two pure subsemimodules of M. Then $IM \cap N = IN$ and $IM \cap L = IL$. Hence $IM \cap (N \cap L) = (IM \cap N) \cap L = IN \cap L$, also $IM \cap (N \cap L) = (IM \cap L) \cap N = IL \cap N$. Hence $IN \cap L = IL \cap N$. On the other hand, $I(N \cap L) = IN \cap IL$. Claim that $IN \cap L = IN \cap IL$. Let $x \in IN \cap L = IL \cap N$. Hence $x \in IN \cap IL$ so $IN \cap L \subseteq IN \cap IL$ and $IN \cap IL \subseteq IN \cap L$. So $IN \cap IL = IN \cap L$. Thus $IM \cap (N \cap L) = IN \cap L = IN \cap IL = I(N \cap L)$. Therefore M is satisfies PIP .

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