Design and Analysis of a New Three-dimensional Hyper Chaotic System

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Abstract

This paper introduces a new three-dimensional autonomous hyper chaotic system, where the new system has six positive parameters. The essential properties and dynamic behavior of the new system are examined with existence of chaotic attractor, dissipativity, symmetry, equilibrium points, Lyapunov Exponents, Kaplan-Yorke dimension, waveform analysis and sensitivity toward initial conditions. The results of the analysis show that the new system has two unstable equilibrium points, Maximum Lyapunov Exponent (MLE) is obtained as 1.27325 and Kaplan-Yorke dimension is obtained as 2.39185, the waveform of the new system is non-periodic and it has high sensitivity towards initial conditions.

Keyword: chaotic, hyper, three-dimensional, unstable, waveform analysis, sensitivity to initial conditions.

المستخلص

يقدم هذا البحث نظام فوضوي جديد مفرط ثلاثي الأبعاد، النظام الجديد يحتوي على ستة مَعلمات موجبة. تم التحقق من الخصائص الاساسية والسلوك الديناميكي للنظام الفوضوي من خلال أستخدام تحليل التبدد، التماثل، نقاط التوازن، حساب Lyapunov Exponents، حساب بُعد Kaplan-Yorke، وجود الجواذب الفوضوية، الشكل الموجي والحساسية للشروط الابتدائية. نتائج تحليل النظام تبين ان النظام الجديد يمتلك نقطتي توازن غير مستقرة، قيمة Maximum Lyapunov Exponent تساوي 1.27325 و بُعد Kaplan-Yorke يساوي 2.39185 ، ومن الشكل الموجي للنظام الجديد تبين أنه لا دوري ويمتلك حساسية عالية للشروط الابتدائية.

الكلمات المفتاحية: فوضوي، مفرط، ثلاثي الأبعاد، غير مستقر، تحليل الشكل الموجي، حساسية للشروط الابتدائية.

1. Overview

A chaotic system is a special dynamical nonlinear system, which has a number of characteristics like sensitivity to initial conditions, irregular and unpredictable behavior [1]. When a dynamical ssystem evolves with particular values of initial conditions and parameters, the chaotic behavior is obtained. In 1963, Professor Lorenz discovered by accident the first chaotic system when he built a model for weather modeling, the model of the system is created via three first order differential autonomous equations [2]. Various three-dimensional chaotic systems are presented such as LÜ system, Chen system, Rossler

system, etc. [3]. Due to Chaos properties, it is implemented nowadays in many various areas like engineering, computer science, mathematics, physics, geology, biology, robotics and etc. [4]. A new three-dimensional chaotic system is presented in this paper and the dynamical behavior properties of the new system is examined using Mathematica programming language.

2. Construction of New Three-dimensional Chaotic System

A mathematical model represents a new 3-D hyper chaotic system constructed via the following first order differential equation shown in Equation (1):

$$\begin{cases} \frac{dx}{dt} = ay - bx \\ \frac{dy}{dt} = -cxz \\ \frac{dz}{dt} = -d + exy + f\sin(y) \end{cases} \quad \ldots(1)$$

In which (x , y , z and t) ∊ R and called the system states, where the constant parameters are

(a , b , c , d , e and f). System (1) shows chaotic behavior when choosing the parameters as: (a= 9, b=2, c= 0.2, d= 42, e= 1.1, f= 33) and initial conditions as: (x (0) = −0.07, y (0) = 5 and z (0) = 0), The strange attractors of system (1) in 3-D view are shown in Figures 1–6 and the strange attractors in 2-D view are shown in Figures 7-12.

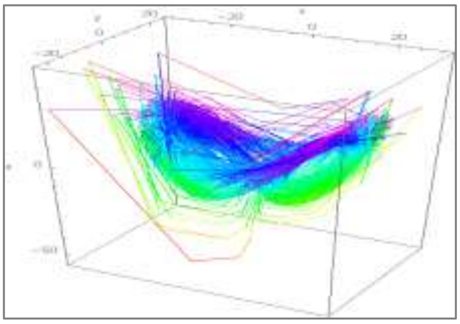

Figure 1. Attractor in (*x*, *y*, *z*) Of new system.

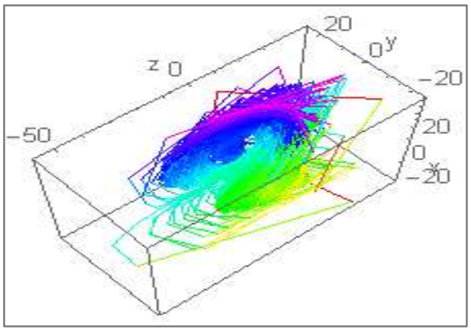

Figure 2. Attractor in (*x*, *z*, *y*) Of new system.

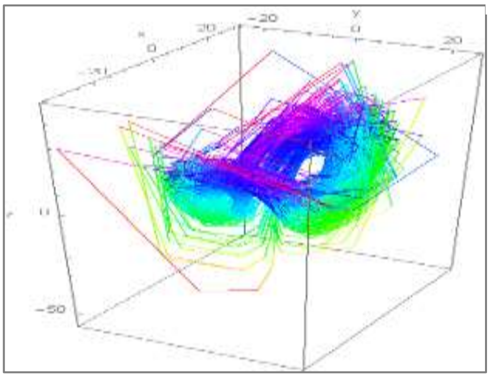

Figure 3. Attractor in (*y*, *x*, *z*) Of new system.

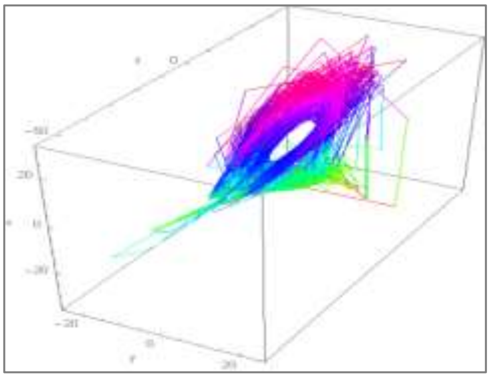

Figure 4. Attractor in (*y*, *z*, *x*) Of new system.

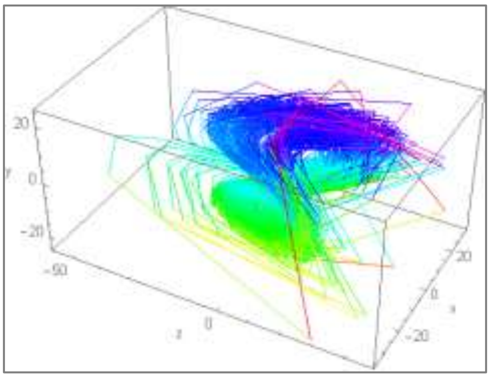

Figure 5. Attractor in (*z*, *x*, *y*) Of new system.

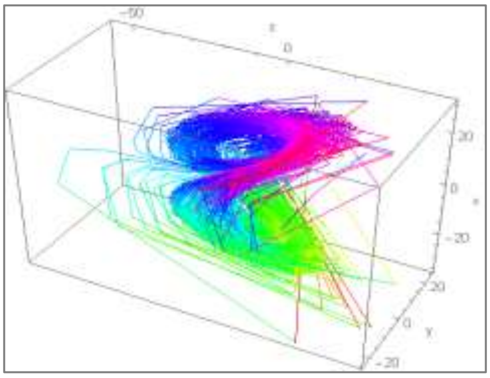

Figure 6. Attractor in (*z*, *y*, *x*) Of new system.

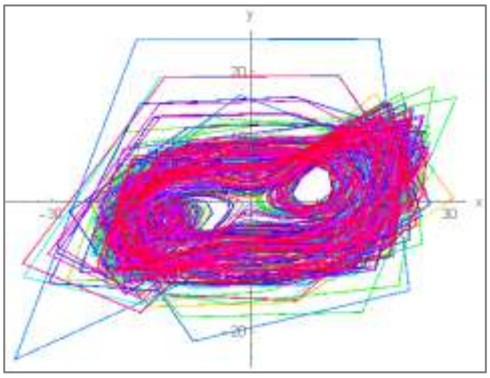

Figure 7. Attractor in (*x*, *y*) of new system.

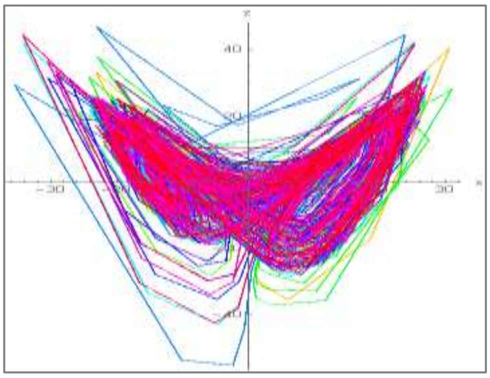

Figure 8. Attractor in (*x*, *z*) of new system.

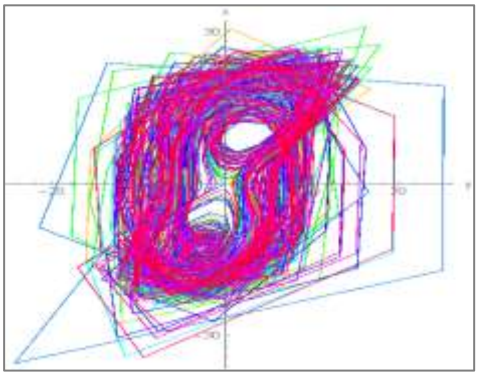

Figure 9. Attractor in (*y*, *x*) of newsystem.

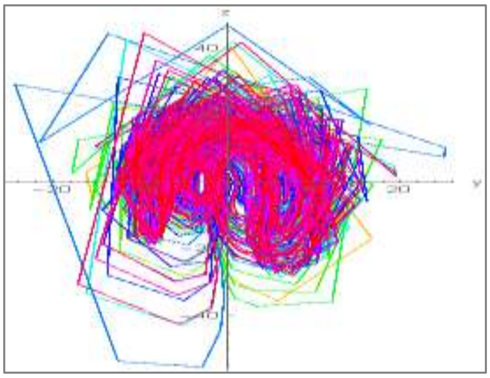

Figure 10. Attractor in (*y*, *z*) of new system.

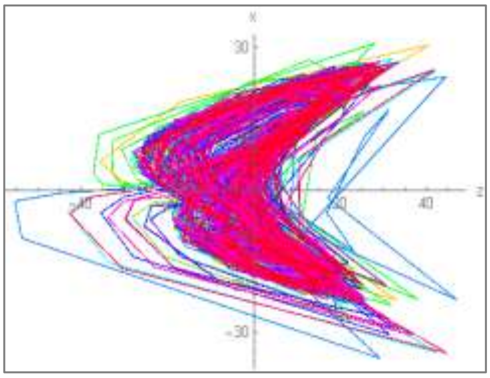

Figure 11. Attractor in (*z*, *x*) of new system

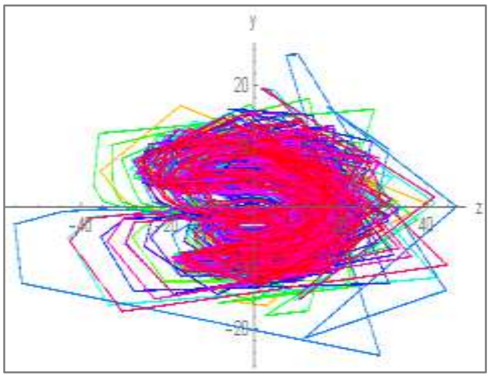

Figure 12. Attractor in (*z*, *y*) of new system.

3. Dynamic Analysis of New Chaotic System

The essential and complex dynamic behavior of the new system are investigated in this section and it has the following basic characteristics:

3.1 Dissipiativity

When system (1) is described in vector formula as shown in Equation (2):

$$f = \begin{cases} f_1 = \dfrac{dx}{dt} = ay - bx \\ f_2 = \dfrac{dy}{dt} = -cxz \\ f_3 = \dfrac{dz}{dt} = -d + exy + f\sin(y). \end{cases} \quad \dots(2)$$

And chosen the control parameters as: (a = 9, b =2, c = 0.2, d = 42, e = 1.1, f = 33).

Equation (3) shows the divergence obtained for vector field f on R3 as follows:

$$\nabla . f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -(b + 0 + 0) < 0 \quad \dots(3)$$

Where $(b + 0 + 0) > 0$, let D stands for any region in R3 with a smooth boundary,

D (t) = $\Omega_t(t)$, such that Ω_t stands for the flow of vector field f and V(t) indicates to volume of D (t), via Liouville's theory, the following formulae is obtained in Equation (4):

$$\frac{dV}{dt} = \int_{D(t)} (\nabla . f)\, dx\, dy\, dz \qquad \dots (4)$$

When the value of $\nabla . f$ is substituted in (4), the following equation is obtained:

$$\frac{dV}{dt} = (-b)V(t) \qquad \dots (5)$$

Then, the first order differential equation (5) is solved and the following unique solution is obtained as:

$$V(t) = V(0)e^{(-b)t} \qquad \dots \ (6)$$

From (6), any volume $V(t)$ should be shrinking fast to zzero exponentially with time as $\rightarrow \infty$, accordingly, new system (1) is considered as a dissipative system.

3.2 symmetry

System (1) has symmetry and invariant around z−axis when the coordinate (x, y, z) is transformed into (−x, −y, z), in order to demonstrate that, let:

$$x = -x, y = -y, z = z \qquad \dots (7)$$

Then, the following equations are obtained:

$$-\frac{dx}{dt} = \frac{dx}{dt}, -\frac{dy}{dt} = \frac{dy}{dt} \ and \ \frac{dz}{dt} = \frac{dz}{dt} \qquad \dots (8)$$

According to Equations (7) and (8), the result is:

$$-\frac{dx}{dt} = ay - bx$$
$$-\frac{dy}{dt} = -cxz \qquad \dots (9)$$
$$\frac{dz}{dt} = -d + exy + fsin(y)$$

$\Rightarrow$

$$\frac{dx}{dt} = -ay + bx$$
$$\frac{dy}{dt} = cxz \qquad \dots (10)$$
$$\frac{dz}{dt} = -d + exy + fsin(y)$$

It could be easily realized that the new system (1) has symmetry around z–axis. For the flow of new system (1), z−axis is invariant and for all values of time the whole orbits of system (1) beginning from the z−axis remain in the z−axis [6].

3.3 Equilibrium Point

In order to find the equilibria of system (1), the nonlinear equations must be solved as follows:

$$ay - bx = 0$$
$$-cxz = 0 \qquad \dots (11)$$
$$-d + exy + fsin(y) = 0$$

And parameters of system (1) are selected as:

a = 9, b =2, c = 0.2, d = 42, e = 1.1, f = 33 ... (12)

Solving Equation (11) with (12), two equilibrium points of system (1) are gained as below:

E1={ $x = 1.47706$, $y = 1.12526$, $z = 0.43637$ }

E2={ $x = -0.64364$, $y = -0.54342$, $z = 0.57381$ }

The new system (1) is linearized around E^* equilibrium point, in order to determine the Jacobian matrix, the following formulae is obtained:

$$J(E^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} \quad \dots (13)$$

$$J(E^*) = \begin{bmatrix} -b & a & 0 \\ -cz & 0 & -cx \\ ey & ex + f\cos(y) & 0 \end{bmatrix} \dots (14)$$

Now, in order to confirm its unitability, the Jacobian matrix is found at each equilibrium point. Equation (15) shows the Jacobean matrix at equilibrium pointE_1:

$$J_1 = J(E_1) = \begin{bmatrix} -2 & 9 & 0 \\ -0.087274 & 0 & -0.295412 \\ 1.237786 & 15.8528 & 0 \end{bmatrix} \dots (15)$$

To gain the eigenvalues, let $[\lambda_I - J1] = 0$, then the matrix J_1 has three eigenvalues as:

$\lambda_1 = 0.0845841+2.41403i$, $\lambda_2 = 0.0845841-2.41403i$ and $\lambda_3 = -2.16917$.

The resulting eigenvalues are real numbers and E_1 is considered as saddle point, so, it could be considered as unstable.

Next, the Jacobian matrix of equilibrium point E_2 is obtained as:

$$J_2 = J(E_2) = \begin{bmatrix} -2 & 9 & 0 \\ -0.114762 & 0 & -0.128728 \\ 0.597762 & 27.5461 & 0 \end{bmatrix} \quad \ldots (16)$$

In the same way, the eigenvalues of matrix J_2 are obtained as:

λ_1= -1.82646, λ_2 = 0.0867681+2.06241^i and λ_3= 0.0867681-2.06241^i

Due to the result from eigenvalues, E_2 is considered a saddle point and unstable [7]. Hence, all equilibria of system (1) are unstable.

3.4 Lyapunov Exponents and Lyapunov Dimension

Lyapunov Exponents of system (1) are obtained as: LE1 =1.27325, LE2 = 0.01566 and LE3 = −3.28925 when control parameters of system (1) are selected as: (a = 9, b =2, c = 0.2, d= 42, e= 1.1, f= 33) and set the initial conditions as (x (0) = −0.07, y (0) = 5 and z (0) = 0), where Maximal Lyapunov Exponent (MLE) is positive which is LE1 =1.27325, indicating the chaos characteristics of the new chaotic system. The new system is hyper chaotic since it has two positive values of Lyapunov Exponents. Since LE1+LE2 > 0 and LE1+LE2+LE3 < 0, the Lyapunov dimension or Kaplan−Yorke dimension of the new system could be obtained as follows [8]:

$$D_{KY} = 2 + \frac{1}{|L_{j+1}|} \sum_{i=1}^{2} L_i$$

$$= 2 + \frac{L_1 + L_2}{|L_3|}$$

$$= 2 + \frac{1.27325 + 0.01566}{3.28925}$$

$$= 2.39185.$$

Kaplan−Yorke dimension of the new system is fractional and this fractional nature referred that the new system has non−periodic orbits and its nearby trajectories diverge. So, there is chaos in this chaotic system.

3.5 Waveform Analysis

The waveforms of system (1) for (x (t), y (t), z (t)) in time domain are displayed in Figures 5–7. Obviously, the time domain waveform has non−cyclic feature, which is one of chaotic system characteristics.

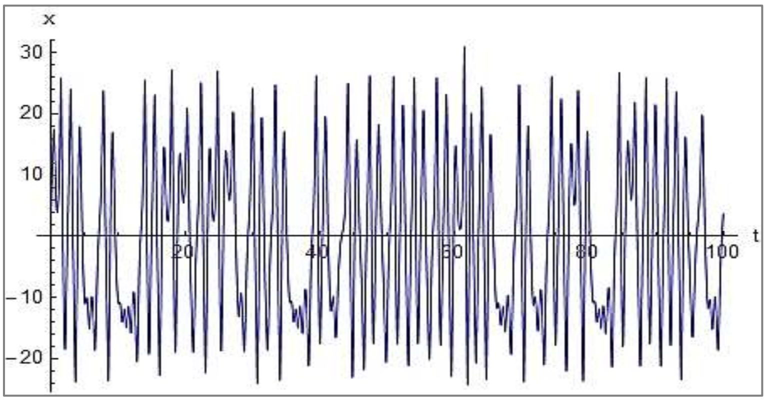

Figure 5. Time vversus x of new system.

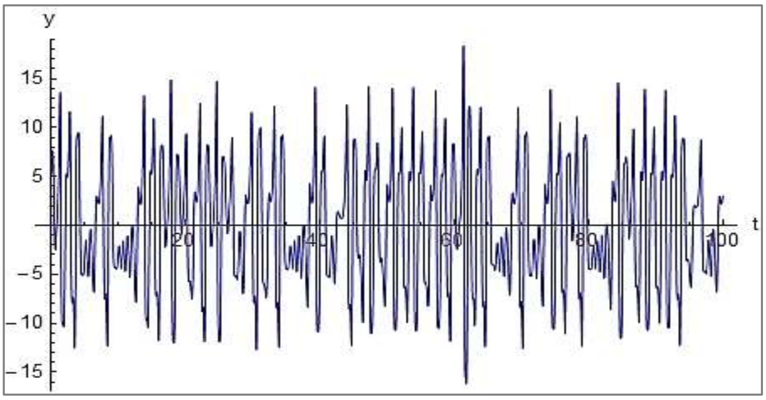

Figure 6. Time versuss y of new system.

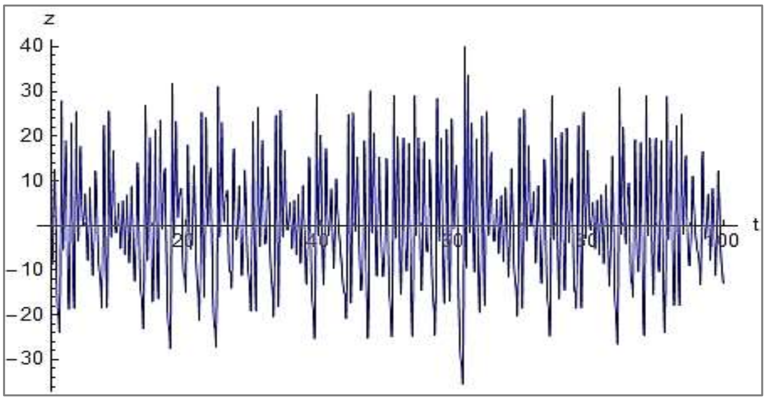

Figure 7. Time versuss z of new system.

3.6 Sensitivity to Initial Conditionss

The long term unpredictability is one of the chaotic system characteristic because of sensitive dependence to initial conditions, such that if a small change is happened between two initial cconditions will become widely separated and the way in which the system is evaluated cannot be predicted [9]. Figures 16 −18, demonstrate the evolution of the chaotic trajectories has high sensitivity towards initial conditions. Here, initial values of system (1) are x (0) = − 0.07,

y (0)= 5, z (0)=0 for solid line and x (0)=−0.07, y (0)=5.00000001, z (0)=0 for dashed line.

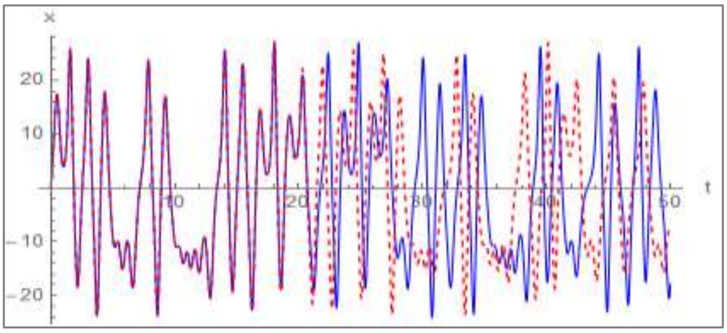

Fig. 16. New system x (t) Sensitivity tests.

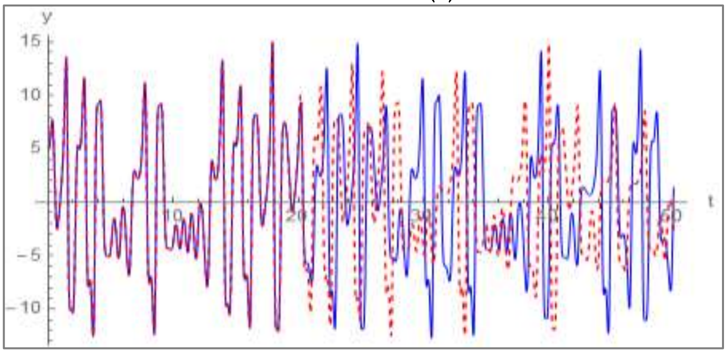

Fig. 17. New system y (t) Sensitivity tests.

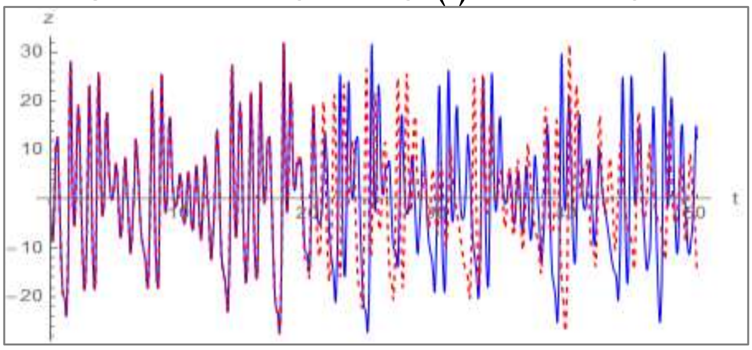

Fig. 18. New system z (t) Sensitivity tests.

4. Conclusion

In this paper introduced a new 3-D hyper chaotic system. The essential properties and dynamical behavior of the new system are examined to prove it's chaotic. The new system generates chaotic behavior when selected the parameters as: a = 9, b =2, c = 0.2, d = 42, e= 1.1, f= 33 and initial conditions are set as: x (0) = −0.07, y (0) = 5 and z (0) =0, the Lyapunov Exponents of the new system are obtained as: LE1 = 1.27325, LE2 =

0.01566 and LE3 = −3.28925, which means the system is hyper chaotic because two positive Lyapunov Exponents are obtained for the new system, the new system has two unstable equilibrium points, the fractal dimension is 2.39185 and the new system characterize with high sensitivity to initial condition and generates complex chaotic attractor. The new chaotic system is suitable to be used in numerous applications and could be employed for information encryption.

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