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Triple Operators of Order n on a Hilbert Space

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Abstract

In this paper, we introduce a new class of operators on a complex Hilbert space $\mathcal{H}$ which is called triple operators of order n. An operator $T \in \mathcal{B}(\mathcal{H})$ is called triple operator of order n if $(T^nT^*)T = T(T^nT^*)$ $for\ all\ \ n \geq 2$. where T^* is the adjoint of the operator T.
We investigate some basic properties of such operators and study the relation between the triple operators of order n and some kinds of operators.

1. Introduction

Functional analysis is a generation and combination of linear algebra, analysis and geometry expressed in simple mathematical notion which allows these aspects of the considered problem to be easily seen. It recognizes because it has the buck up of vast mathematical machinery which subsumes many of classical results on differential equations, analysis, numerical method and applied mathematical techniques [1].
One of the important notions in applied mathematics and systems analysis is the operator theory investigation by obtaining a mathematical model, and then determining such properties as existence, uniqueness and regularity of solutions[2]. Thus, let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called idempotent operator if $T^2 = T$ [3], self-adjoint if $T^* = T$ [4], unitary if $T^*T = TT^* = I$ and partial isometry if $TT^*T = T$ [5].The operator $T \in \mathcal{B}(\mathcal{H})$ is said to be nilpotent operator if $T^n = 0$ [6] and isometry if $T^*T = I$ [7].

2- Triple operators of order n

In this section, we will introduce and study some essentially properties for the triple operators of order n.

Definition (2.1): The operator $T \in \mathcal{B}(\mathcal{H})$ is called triple operator of order n if $(T^nT^*)T = T(T^nT^*)$, where n is positive integer number greater than or equal 2 and T^* is the adjoint of the operator T.

Example (2.2): Let $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ be an operator on two-dimensional Hilbert space $\mathbb{C}^2$ .Then $(T^2T^*)T = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} = T(T^2T^*)$
Therefore T is triple operator of order 2.

Example (2.3): If $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is an operator on two-dimensional Hilbert space $\mathbb{C}^2$,Then $(T^2T^*)T = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \neq T(T^2T^*) = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$
Thus T is not triple operators of order 2 .
In the following proposition, we give some properties of the triple operators of order n.

Proposition (2.4): If T is triple operator of order n on a Hilbert space $\mathcal{H}$. Then:
1- kT is triple operator of order n for every complex number k.
2- If S is unitarily equivalence to T, then S is triple operator of order n.
3- If M is closed subspace of $\mathcal{H}$, then (T/M) is triple operator of order n .

Proof: (1) $[(kT)^n(kT)^*](kT) = (k^nT^n\overline{k}T^*)kT = k^n\overline{k}\,k(T^nT^*)T$
$= kk^n\overline{k}\,T(T^n\,T^*) = (kT)[(kT)^n(kT)^*]$
So that (kT) is Triple operator of order n .
(2) since S is unitarily equivalence to T. Then there exists unitary operator U such that $S = UTU^*$, so that $S^* = UT^*U^*$ and $S^n = UT^nU^*$

$(S^n S^*)S = (UT^nU^*UT^*U^*)UTU^* = U(T^nT^*)TU^*$
Since T is triple operator of order n, then $(S^n S^*)S = UT(T^nT^*)U^*$
On the other hand $S(S^n S^*) = UTU^*(UT^nU^*UT^*U^*) = UT(T^nT^*)U^*$
Thus S is triple operator of order n.

(3) $[(T/M)^n(T/M)^*](T/M) = [(T^n/M)(T^*/M)](T/M) = (T^nT^*/M)(T/M) = [(T^nT^*)T]/M = T(T^nT^*)/M$
$= (T/M)[(T^nT^*)/M] = (T/M)[(T/M)^n(T/M)^*]$
Therefore T/M is triple operator of order n.

Remark (2.5): If T is triple operator of order n, then not necessary T^* is triple operator as we saw in the following example:

Example (2.6): Let U^* be the adjoint of the unilateral shift operator on ℓ_2. (i.e. $U^*(x_1,x_2,x_3,\dots) = (x_2,x_3,x_4 \dots)$)
Since $(U^{*n}U)U^* = U^{*n} = U^*(U^{*n}U)$. Then U^* is triple operator of order n.
But $(U^nU^*)U = U^n \neq U(U^nU^*) = U^{n+1}U^*$
so that U is not triple operator of order n.

The following example shows that if S,T are triple operators of order n, then not necessary $(S+T)$ *and* $(S.T)$ are triple operators of order n.

Example (2.7): Let $S = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ and $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ are operators on two dimensional Hilbert space $\mathbb{C}^2$.

Since $(S^2S^*)S = \begin{pmatrix} 81 & 0 \\ 0 & 256 \end{pmatrix} = S(S^2S^*)$ and $(T^2T^*)T = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} = T(T^2T^*)$

Then S and T are triple operators of order 2.
But $[(S+T)^2(S+T)^*](S+T) = \begin{pmatrix} 11 & -12 \\ 2 & 29 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 10 & -47 \\ 33 & 85 \end{pmatrix} \neq$
$(S+T)[(S+T)^2(S+T)^*] = \begin{pmatrix} 20 & -53 \\ 17 & 75 \end{pmatrix}$ and
$[(ST)^2(ST)^*](ST) = \begin{pmatrix} 54 & -96 \\ 72 & 128 \end{pmatrix}\begin{pmatrix} -3 & -3 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} -222 & 546 \\ 296 & -728 \end{pmatrix} \neq (ST)[(ST)^2(ST)^* = \begin{pmatrix} -54 & -96 \\ -504 & -896 \end{pmatrix}$
Therefore $(S+T)$ *and* (ST) are not triple operators of order 2.

The following theorem show that the condition on S,T is very necessary to becomes $(S+T)$ is triple operators of order n.

Theorem(2.8): If S,T are commuting triple operators of order n such that $T^n = S^n = 0$, then $(S+T)$ is triple operator of order n.

Proof: To prove that $[(S+T)^n(S+T)^*](S+T) = (S+T)[(S+T)^n(S+T)^*]$
Thus $[(S+T)^n(S+T)^*](S+T) = [(S^n+T^n)(S^*+T^*)](S+T)$
$= (S^n S^*)S + (S^nT^*)S + (T^nS^*)S + (T^nT^*)S + (S^nS^*)T + (S^nT^*)T + (T^nS^*)T + (T^nT^*)T$
Since S and T are triple operators of order n and $T^n = S^n = 0$,then
$[(S+T)^n(S+T)^*](S+T) = S(S^nS^*) + T(T^nT^*) = (S+T)[(S+T)^n(S+T)^*]$
Thus $(S+T)$ is triple operator of order n.

The following theorem show that the condition of S,T is very necessary to becomes (ST) is triple operators of order n .

Theorem (2.9): If T is triple operator of order n and S is isometry operator ,
such that (1) $ST = TS$ (2)$S^nT = TS^n$, Then (ST) is triple operator of order n.

Proof: since S is isometry operator ,then $S^*S = I$
$[(ST)^n(ST)^*](ST) = [S^nT^nT^*S^*](ST) = S^n(T^nT^*)T = S^nT(T^nT^*)$
Multiplying of the left side by S and right side by S^* we get
$= SS^nT(T^nT^*)S^* = ST(S^nT^nT^*S^*) = (ST)[(ST)^n(ST)^*]$
Therefore (ST) is triple operator of order n.

Theorem (2.10): If $T_1,T_2,\dots,T_m$ are triple operators of order n , Then the direct sum and the tensor product are triple operators of order n.

Proof:
$[(T_1\oplus T_2\oplus \dots\oplus T_m)^n(T_1\oplus T_2\oplus \dots\oplus T_m)^*](T_1\oplus T_2\oplus \dots\oplus T_m)$
$= [(T_1{}^n\oplus T_2{}^n\oplus \dots\oplus T_m{}^n)(T_1^*\oplus T_2{}^*\oplus \dots\oplus T_m^*)](T_1\oplus T_2\oplus \dots\oplus T_m)$
$= (T_1{}^nT_1^*\oplus T_2{}^nT_2{}^*\oplus \dots\oplus T_m{}^nT_m^*)(T_1\oplus T_2\oplus \dots\oplus T_m)$
$= (T_1{}^nT_1^*)T_1\oplus (T_2{}^nT_2{}^*)T_2\oplus \dots\oplus (T_m{}^nT_m^*)T_m)$
Since $T_1,T_2,\dots,T_m$ are triple operators of order n. Then
$= T_1(T_1{}^nT_1{}^*)\oplus T_2(T_2{}^nT_2{}^*)\oplus \dots\oplus T_m(T_m{}^nT_m^*))$
$= (T_1\oplus T_2\oplus \dots\oplus T_m)[(T_1\oplus T_2\oplus \dots\oplus T_m)^n(T_1\oplus T_2\oplus \dots\oplus T_m)^*]$
Also $[(T_1\otimes T_2\otimes \dots\otimes T_m)^n(T_1\otimes T_2\otimes \dots\otimes T_m)^*](T_1\otimes T_2\otimes \dots\otimes T_m)$
$= [(T^n{}_1\otimes T_2{}^n\otimes \dots\otimes T_m{}^n)(T_1^*\otimes T_2{}^*\otimes \dots\otimes T_m^*)](T_1\otimes T_2\otimes \dots\otimes T_m)$
$= (T_1{}^nT_1^*\otimes T_2{}^nT_2{}^*\otimes \dots\otimes T_m{}^nT_m^*)(T_1\otimes T_2\otimes \dots\otimes T_m)$
$= ((T_1{}^nT_1^*)T_1\otimes (T_2{}^nT_2{}^*)T_2\otimes \dots\otimes (T_m{}^nT_m^*)T_m)$
Since $T_1,T_2,\dots,T_m$ are triple operators of order n. Then
$= (T_1(T_1{}^nT_1^*)\otimes (T_2(T_2{}^nT_2^*)\otimes \dots\otimes (T_m(T_m{}^nT_m^*))$
$= (T_1\otimes T_2\otimes \dots\otimes T_m)[(T_1\otimes T_2\otimes \dots\otimes T_m)^n(T_1\otimes T_2\otimes \dots\otimes T_m)^*]$

Theorem (2.11): If T is triple operator of order n, then T is triple operator of order $(n+1)$.

Proof: since T is triple operator of order n, then $(T^nT^*)T = T(T^nT^*)$
$\Rightarrow (T^{n+1}T^*)T = T(T^{n+1}T^*)$
The converse of the theorem (2.11) is not true as the following example :

Example (2.12): If $T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$ be operator on a Hilbert space $\mathbb{C}^3$.

Then $(T^3T^*)T = T(T^3T^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

But $(T^2T^*)T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \neq T(T^2T^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Thus T is triple operator of order (3) but not triple of order (2).

3- Relation between triple operator of order n and some other kinds of operators

Proposition (3.1): 1- Every isometry operator is triple operator of order n.

2- Every self adjiont operator is triple operator of order n .

3- Every unitary operator is triple operator of order n .

4- If T is nilpotent operator , then T is triple operator of order n .

The following two examples shows Triple operators of order n and partial isometry operator are independent .

Example (3.2): If $T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ operator on a Hilbert space $\mathbb{C}^2$,then

$TT^*T = \begin{pmatrix} 4 & 4 \\ -4 & -4 \end{pmatrix} \neq T \Rightarrow T$ is not partial isometry operator.

But $(T^2T^*)T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = T(T^2T^*)$.Therefore T is triple operator of order 2.

Example (3.3): Let $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ be an operator on a Hilbert space $\mathbb{C}^3$. Then $TT^*T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = T \Rightarrow T$ is a partial isometry.

But $(T^2T^*)T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq T(T^2T^*)$.Thus T is not triple operator of order 2.

Remark (3.4): The class of all idempotent operator and the class of triple operators of order n are independent, as we saw in the following examples:

Example (3.5): Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ be an operator on a Hilbert space $\mathbb{C}^2$.Then

$T^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = T \Longrightarrow T \text{ is idempotent operator}$.

But $(T^2T^*)T = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \neq T(T^2T^*) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Therefore T is not triple operator of order 2 .

Example (3.6): If $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ operator on a Hilbert space $\mathbb{C}^2$, then T is triple operator of order 2 but T is not idempotent operator .

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المؤثرات الثلاثية من الرتبة n في فضاء هلبرت

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الملخص

في هذا البحث سوف نقدم نوع جديد من المؤثرات المعرفة على فضاء هلبرت المعقد الذي أطلقنا عليه اسم المؤثرات الثلاثية من الرتبة n. المؤثر $T \in \mathcal{B}(\mathcal{H})$ يسمى المؤثر الثلاثي من الرتبة n اذا كان $(T^nT^*)T = T(T^nT^*)$ $\forall\, n \geq 2$ ، حيث T^* هو المؤثر المرافق (المصاحب) للمؤثر T .

ونقوم بدراسة بعض الخواص الأساسية لهذا المؤثر والعلاقة بين المؤثرات الثلاثية من الرتبة n مع بعض الأنواع الأخرى من المؤثرات.

الكلمات المفتاحية: المؤثرات الثلاثية، المؤثرات، فضاء هلبرت .