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Polynomials Over Splitting Fields

Majid Mohammed Abid
University of Anbar - College of education for pure sciences .
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Abstract :- In this paper we study some results concerning the existence of splitting fields which are generated by roots of polynomials. Also we study the roots of cubic polynomials.

Key words : Polynomials , Over Splitting Fields

Introduction and preliminaries

These results are basic to Galois theory consider the polynomial ring $K[X]$ over field K .Let f(x) belong to K[X] in the quotient ring K[X]/f(x). We let g(x) denotes the coset (g(x)+f(x)). Thus if $g(x)=\sum_{i=0}^{n} K_i x^i$, then by the definition of addition and multiplication of cosets we have that $\overline{g(x)}=\sum_{i=0}^{n} \bar{K}_i x^{\bar{i}}$, we considered a field K contains in a complex numbers $\mathbb{C}$ and a cubic polynomial $f(x)=x^3+px+q\in K[X]$. Also, we obtained explicit expression involving extraction of square and cubic roots for the three roots α_1,α_2 and α_3 of f(x) in C and we were beginning to study the splitting field extension $E=K(\alpha_1,\alpha_2.\alpha_3)$ .If f(x) factors in $K[X]$ either all the roots are in K or exactly one of them (say α_3) is in K and the other two roots of irreducible quadratic polynomial in $K[X]$ In this case $E=K(\alpha_1)$ is a field extension of dimension 2 over K. Therefore if α_1 denotes one of the roots, we know that $K(\alpha_1)\cong K(X)/(f(X))$ is a field extension of dimension 3 =deg(f)over K .also we have $K\in K(\alpha_1)\subseteq E$, it follows from the multiplicatively of dimension that 3 divides the dimension of E over K.

Definition. [2]

A polynomial f(x) belong to K[X] is said to split over a field S contains K,if f(x) can be write it factor as product of linear a factors in S[X],such that K is a field.

Remark .[1]

$\delta=(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)\in E$,since $\delta^2=-4p^3-27q^2\in K$,either K or K(δ) is an extension field of dimension 2 over K, since $K\subseteq K(\delta)\subseteq E$ it follows that 2 also divides $\dim_k$ (E).

$\delta\in K$ and $\dim_k$ (E) =3 or

$\delta\notin K$. and $\dim_k$ (E) =6.

Proposition. [4]

Let K be a field .If f(x) is a non-constant polynomial in $K[X]$, then there exists a field extension F/K such that F contains a root of f(x) .

Now by the following we can show that C is the field of complex numbers [x^2+1 is irreducible in $R[X]$. Now, $R[X]=\{a+b\bar{x}|\ a,b\in R\}$ is a field where $\bar{x}=x+(x^2+1)$ .Since $x^2=-1$, we may call C the field of the complex numbers.]

Definition . [5]

Let K be a field .A polynomial $f(x)\in K[X]$ is said to split over a field S$\supseteq$K if f(x) can be factored as a product of line a factors in S[x].

A field S containing K is said to be a splitting field for f(x) over K if f(x) splits over S but over no proper intermediate field of S/K.For example The field of complex numbers C is s splitting field for the polynomial x^2+1 over R .this follows, since $x^2+1=(x+i)(x-i)$ in C[x], and C/R has no proper intermediate field because [C:R]=2 .Now if $C \supseteq L \supseteq R$ where L is an intermediate field of C/R, then 2=[C:R]] [L:R] and so either [C:L][=1 or [L:R]=1 .Then either C=L or C=R and note that C is the splitting field of x^2+1 over Q since x^2+1 splits over Q (L).

Proposition . [5]

Let K be a field and f(x) be a polynomial in $K[X]$ of degree n . Let F/K be a field extension. If f(x)=c(x-c1)(x-c2)….(x-cn) in F(x). then `is a splitting field for f(x) over K.

Also,if we have K a finite field.Then cardinality of K is pn for some prime p and some positive integer n.Every k belong to K is a root of the polynomial XPn –X and K is the splitting field of this polynomial over prime subfield Zp.

Therefore,if the roots are known as α1 and α2 then The field $Q(\lambda, \lambda_3)$ for the last example is a splitting field for x^4-3 over Q.

Now we can say that if K be field and f(x) be constant polynomial over K. Then there is a splitting field for f(x) over K. and if E /K is a field extension and f(x) be an irreducible polynomial in $K[X]$. If $a,b \in E$ are roots of f(x) then $K(a) \cong K(b)$.

Also, we can use other concept to obtain splitting field by normal extension such that ((if a finite extension E / K is normal ,then it is a splitting field over K and f(x) bolong to K[X].)).

Therefore , if E / L and L / K be a finite extensions and if E / K is normal then E / L is normal(E /L is splitting).Now we can give the following fact about two splitting fields[Let $f(x) \in K[x]$. Any two splitting fields for f(x) over K are isomorphic],

also, let F/K be a field extension and $a,b \in F$. Then a and b are called conjugates, if a and b are roots of the same irreducible polynomial over K.

Examples

1-The field $Q(\sqrt{2}) = \{a+b\sqrt{2} : a,b \in Q\}$ is a splitting field of $x^2-2 \in Q[x]$ over Q

2- A splitting field of $x^2+1 \in R[x]$ over R is the field C.

Proposition[2]

If K is field and $f \in K[x]$ then:

There exists splitting field of polynomial; f on K.

Any two splitting fields of f on K are two isomorphism fields on K.

Splitting fields are unique up to isomorphism over K.

Proposition .[3]

Let K be subfield of C let $f(x)=x^3+px+q \in K[X]$ an irreducible cubic polynomial and let E denotes the splitting field of f(x) in C. Let $\delta=(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)$ where α_i are the roots of f(x). If $\delta \in K$, then $\dim_k$ (E)=6

Proposition. [1]

Suppose $K \subseteq L$ is any field extension f(x) $\in K[X]$ and β is the root of f(x) in L. If δ is an automorphism of L leaves F fixed pointwise, then $\delta(\beta)$ is also a root of f(x).

Proof

If $f(x)=\sum f_i x^i$,and since β is one of the roots that is mean f(β)=0 then $\sum f_i \delta(\beta)^i = \delta \sum f_i(\beta^i) = \delta(0)=0$

Example

Let $f(x)=x^3-2$,which is irreducible over Q. The three roots of f in C are $\sqrt[3]{2}$, $\omega\sqrt[3]{2}$ and $\omega^2\sqrt[3]{2}$, where $\omega=\frac{1}{2}+\frac{\sqrt{-3}}{2}$ is a primitive cube root of 1.

Finally,to show that the splitting fields always exist[for if g is any irreducible factor of f,then K[X]/ (g)=K(α) is an extension of K for which g(α)=0,where α denotes the image of X.Then g and f are splits off a linear factor,induction implies that exists a splitting field L for f.

Conclusions

We gote that a polynomial f(x) $\in K[X]$ always has a splitting field, namely the field generated by its roots in a given algabric closure $\overline{K}$ of K. Also we can apply these roots of any non-constant polynomials by Galois theory.We obtained a new result (every normal extension is splitting field, and splitting fields are unique. let K be a field by a root of polynomials f(x) $\in K[X]$ we mean an element α in an over field of K such that $f(\alpha)=0$. It is easy to see that a non-zero polynomial in $K[X]$ of degree n has most n roots.

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متعددات الحدود على الحقل المنفصل

ماجد محمد عبد
Email: scianb@yahoo.com

الخلاصة

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