On Artin Cokernel of the Group $D_{nh}$ When n is an Odd Number

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Abstract

The group of all $Z$-valued characters of $G$ over the group of induced unit characters from all cyclic subgroups of $G$ forms a finite abelian group, called Artin Cokernel of $G$, denoted by $AC(G) = \bar{R}(G)/T(G)$

The problem of finding the cyclic decomposition of Artin cokernel $AC(D_{nh})$ has been considered in this paper when $n$ is an odd number, we find that if $n = p_1^{a_1} p_2^{a_2}... p_m^{a_m}$, where $p_1, p_2,..., p_m$ are distinct primes and not equal to 2, then

$$AC(D_{nh}) = \bigoplus_{i = 1}^{2(\alpha_1+1)(\alpha_2+1)...(\alpha_m+1)-1} C_2$$

$$= \bigoplus_{i = 1}^{2} AC(D_{n}) \bigoplus C_2$$

And we give the general form of Artin's characters table $Ar(D_{nh})$ when $n$ is an odd number.

المستخلص

إن زمرة كل الشواخص العمومية ذات القيم الصحيحة للزمرة $G$ على زمرة الشواخص المحتملة من الشواخص الأحادية للزمرة الجزئية الدائرية من الزمرة $G$ تكون زمرة أبيلية متميزة و تسمى النواة المشترك – أرتن – للزمرة $D_{nh}$، ويرمز لها بالرمز $AC(G) = \bar{R}(G)/T(G)$. إن مسألة إيجاد التجزئة الدائرية لزمرة القسمة $AC(G)$ قد أُعتبرت في هذه الرسالة للزمرة $D_{nh}$ عندما $n$ عدد فردي، فقد

$$n = p_1^{a_1} p_2^{a_2}... p_m^{a_m}$$

حيث إن $a_1, a_2,..., a_m$ أعداد أولية مختلفة لا تساوي 2 فإن:
Introduction:

The abelian group of all $Z$-valued characters of a finite group $G$ under the operation of pointwise addition over the group of induced unit characters form all cyclic subgroups of the group $G$ (Artin characters) form a finite abelian group which is called Artin Cokernel of the group $G$, denoted by $AC(G)$ . The problem of determining the cyclic decomposition of $AC(G)$ seem to be untouched . In this work, $G$ is considered to be the dihedral group $D_{nh}$ when $n$ is an Odd number . To do this work we must do the following steps:

1. We must know the rational valued characters table of the group $D_{nh}$, $\equiv^*(D_{nh})$.
2. We must find Artin characters table of the group $D_{nh}$, $Ar(D_{nh})$.
3. We must find the matrix which expresses the Artin characters of the group $D_{nh}$ in terms of rational valued characters, $M(D_{nh}) = Ar(D_{nh}) . (\equiv^* D_{nh})^{-1}$.
4. From (3) we must find the invariant factors matrix $M(D_{nh})$.
5. From (4) we can find the cyclic decomposition of $AC(D_{nh})$.

The exponent of $AC(G)$ is called the Artin exponent of the group $G$, denoted by $A(G)$ . In 1968 T.Y Lam [15] defined $AC(G)$ and he studied $AC(G)$, when $G$ is acyclic group.


In 2000 H.R.Yassien [6] studied the cyclic decomposition of $AC(G)$ when $G$ is an elementary abelian group . In 2002 H.H.Abbass [5] found $\equiv^*(D_n)$ . In 2006 A.S.Abed [2] found $Ar(C_n)$ when $C_n$ is the cyclic group of order $n$ . In this paper, we find $Ar(D_{nh})$ and we study $AC(D_{nh})$ of the nonabelian group $D_{nh}$, when is an odd number.

1. Some Basic Concepts:-

In this section, we shall give basic concepts, notations and theorems about matrix representation, characters and Artin characters, which will be used in the next sections.

Definition (1.1):[2]

The general Linear group $GL(n,F)$ is a multiplicative group of all non-singular $n \times n$ matrices over the field $F$.

Definition (1.2):[3]

A matrix representation of a group $G$ is a homomorphism of $G$ into $GL(n,F)$, $n$ is called the degree of matrix representation $T$. In particular, $T$ is called a unit representation (principal) if $T(g)=1$, for all $g \in G$.

Definition (1.3):[3]

The trace of an $n \times n$ matrix $A$ is the sum of the main diagonal elements, denoted by $\text{tr}(A)$.
Definition (1.4):[3]
Let $T$ be a matrix representation of degree $n$ of a finite group $G$ over the field $F$. The character $\chi$ of degree $n$ of $T$ is the mapping $\chi : G \rightarrow F$ defined by $\chi(g) = \text{tr}(T(g))$ for all $g \in G$. In particular, the character of the principal representation ($\chi(g) = 1$, for all $g \in G$) is called the principal character.

Definition (1.5):[3]
Two elements $g$ and $h$ in the group $G$ are said to be conjugate if $h = xgx^{-1}$, for some $x \in G$. The relation of conjugacy is an equivalence relation on $G$. The equivalence classes determined by this relation are referred to as the conjugate classes and $C_{L_g}$, $g \in G$ is the conjugate class of the element $g$.

Definition (1.6):[3]
The centralizer of $x$ in $G$ is the subgroup $C_G(x) = \{a \in G : a x a^{-1} = x\}$.

Definition (1.7):[3]
Let $H$ be a subgroup of $G$ and $\phi$ be a character of $H$, the induced character on $G$ is given by

$$
\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^\circ(xgx^{-1})
$$

where $g \in G$ and $\phi^\circ$ is defined by

$$
\phi^\circ(h) = \begin{cases} 
\phi(h) & \text{if } h \in H \\
0 & \text{if } h \notin H 
\end{cases}
$$

Theorem (1.8):[6]
Let $H$ be a cyclic subgroup of $G$ and $h_1, h_2, \ldots, h_m$ are chosen representatives for the $m$-conjugate classes of $H$ contained in $C_{L_g}$, $g \in G$, then

$$
\phi^G(g) = \begin{cases} 
\frac{|C_G(g)|}{|C_H(g)|} \sum_{i=1}^{m} \phi(h_i) & \text{if } h_i \in H \cap C_{L_g} \\
0 & \text{if } H \cap C_{L_g} = \emptyset
\end{cases}
$$

Definition (1.9):[6]
Let $G$ be a finite group, any character induced from the principal character of cyclic subgroup of $G$ is called Artin character of $G$.

Definition (1.10):[9]
Two elements of the group $G$ are said to be $\Gamma$-conjugate if the cyclic subgroups they generate are conjugate in $G$. This defines an equivalence relation on $G$. Its classes are called $\Gamma$-classes.
Proposition (1.11):[15]
The number of all distinct Artin characters on a group G is equal to the number of \( \Gamma \)-classes on G.

Definition (1.12):[2]
The information about Artin characters of a finite group G is displayed in a table called Artin characters table of G, denoted by Ar(G) which is \( l \times l \) matrix whose columns are \( \Gamma \)-classes and rows the values of all Artin characters on G, where \( l \) is the number of \( \Gamma \)-classes.

Definition (1.13):[3]
A rational valued character \( \theta \) of G is a character whose values are in the set of integers \( \mathbb{Z} \), which is \( \theta(g) \in \mathbb{Z} \) for all \( g \in G \).

Proposition (1.14):[12]
The number of all distinct rational valued characters of a finite group G equals the number of \( \Gamma \)-classes on G.

Definition (1.15):[12]
The information about rational valued characters of a finite group G is displayed in a table called the rational valued characters table of G, denoted by \( \equiv^* \( G \) \) which is \( l \times l \) matrix whose columns are \( \Gamma \)-classes and rows are the values of all rational valued characters of G, where \( l \) is the number of \( \Gamma \)-classes.

Theorem [Artin] (1.16):[9]
Every rational valued character of a finite group G can be written as a Linear combination of Artin's characters with coefficient rational numbers.

2. The Factor Group AC(G):

The definition of the factor group AC(G) was introduced by T.Y Lam [15] in 1967. The applications of the factor group AC(G) not only in the mathematics but also in physics and chemistry.

In this section we shall study AC(G), dihedral group \( D_n \) and \( \equiv^* \( D_n \) \), when \( n \) is an odd number.

Definition (2.1):[15]
Let \( \overline{R}(G) \) be the group of \( \mathbb{Z} \)-valued generalized characters of G under the operation pointwise addition and T(G) is the normal subgroup of \( \overline{R}(G) \) generated by Artin's characters.
The abelian factor group \( \overline{R}(G)/T(G) \) is called Artin's Cokernel of G, denoted by AC(G).

Definition (2.2):[12]
Let \( M \) be a matrix with entries in a principle domain \( R \). A \( K \)-minor of \( M \) is the determinant of \( K \times K \) Submatrix preserving row and column order.

Definition (2.3):[12]
A \( K \)-th determinant divisor of \( M \) is the greatest common divisor (g.c.d) of all \( K \)-minor, denoted by \( D_K(M) \).

Theorem (2.4):[12]
Let \( M \) be an \( n \times n \) matrix with entries in a principle domain \( R \), then there exist matrices \( P \) and \( W \) such that
1- \( P \) and \( W \) are invertibles.
2- \( P \cdot M \cdot W = D \).
3- \( D \) is a diagonal matrix.
4- If we denote \( D_{jj} \) by \( d_j \) then there exists a natural number \( m \); \( 0 \leq m \leq n \) such that \( j > m \) implies \( d_j = 0 \) and \( j \leq m \) implies \( d_j \neq 0 \) and \( 1 \leq j \leq m \) implies \( d_j/d_{j+1} \).
Definition (2.5):[12]

Let \( M \) be a matrix with entries in a principal domain \( R \), and equivalent to matrix \( D = \{ d_1, d_2, \ldots, d_m, 0, 0, \ldots, 0 \} \), Such that \( d_j/d_{j+1} \) for \( 1 \leq j \leq m \), \( D \) is called the invariant factor matrix of \( M \) and \( d_1, d_2, \ldots, d_m \) the invariant factors of \( M \).

Remark (2.6):

According to the Artin theorem (1.16) there exists an invertible matrix \( M(G) \) with entries in the field of rational \( \mathbb{Q} \) such that \( \equiv^*(G) = M^{-1}(G) \cdot \text{Ar}(G) \) and this implies \( M(G) = \text{Ar}(G) \cdot (\equiv^*(G))^{-1} \).

By theorem (2.4) there exists two matrices \( P(G) \) and \( W(G) \) such that \( P(G) \cdot M(G) \cdot W(G) = \text{diag} \{ d_1, d_2, \ldots, d_l \} = D(G) \), where \( d_j = \pm D_j(M(G))/D_{j+1}(M(G)) \) and \( l \) is the number of \( \Gamma \)-classes.

Theorem (2.7):[6]

\[
AC(G) = \bigoplus_{j=1}^{l} C_{d_j} \text{ where } d_j = \pm D_j(M(G))/D_{j+1}(M(G)), \quad l \text{ is the number of all distinct } \Gamma \text{-classes and } C_{d_j} \text{ is cyclic subgroup of order } d_j.
\]

Theorem(2.8): [14]

If \( n \) is an odd number such that \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m} \), where \( p_1, p_2, \ldots, p_m \) are distinct primes, then:

\[
(\alpha_1+1)(\alpha_2+1) \cdots (\alpha_m+1) - 1
\]

\[
AC(D_n) = \bigoplus_{i=1}^{l} C_2
\]

Proposition (2.9):[12]

Let \( P \) be a prime number, then the rational valued characters table of cyclic group \( C_p^s = \langle r \rangle \) is given by \( \equiv^*(C_p^s) = \)

<table>
<thead>
<tr>
<th>( \Gamma )-Classes</th>
<th>( [1] )</th>
<th>( [r^{p-1}] )</th>
<th>( [r^{p-2}] )</th>
<th>( [r^{p-3}] )</th>
<th>( \cdots )</th>
<th>( [r^{p}] )</th>
<th>( [r] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0_1 )</td>
<td>( p^{s-1}(p-1) )</td>
<td>( -p^{s-1} )</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 0_2 )</td>
<td>( p^{s-2}(p-1) )</td>
<td>( p^{s-2}(p-1) )</td>
<td>-p^{s-2}</td>
<td>0</td>
<td>( \ldots )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 0_3 )</td>
<td>( p^{s-3}(p-1) )</td>
<td>( p^{s-3}(p-1) )</td>
<td>( p^{s-3}(p-1) )</td>
<td>-p^{s-3}</td>
<td>( \ldots )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( 0_{s-1} )</td>
<td>( p(p-1) )</td>
<td>( p(p-1) )</td>
<td>( p(p-1) )</td>
<td>( p(p-1) )</td>
<td>( \ldots )</td>
<td>( p(p-1) )</td>
<td>-p</td>
</tr>
<tr>
<td>( 0_s )</td>
<td>( p-1 )</td>
<td>( p-1 )</td>
<td>( p-1 )</td>
<td>( p-1 )</td>
<td>( \ldots )</td>
<td>( p-1 )</td>
<td>( p-1 )</td>
</tr>
<tr>
<td>( 0_{s+1} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \ldots )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Remark (2.10) :- In general if \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m} \) where \( p_1, p_2, \ldots, p_m \) are distinct primes, then
\[
\equiv^* (C_n) = \equiv^* (C_{p_1}^{\alpha_1}) \otimes \equiv^* (C_{p_2}^{\alpha_2}) \otimes \cdots \otimes \equiv^* (C_{p_m}^{\alpha_m})
\]
where \( \otimes \) is the tensor product.

Definition (2.12):[9]

The dihedral group \( D_n \) is a certain non-abelian group of order \( 2n \), it is usually thought a group of transformations of Euclidean plane of regular \( n \)-polygon consisting of rotation \( (r^k) \) (about the origin) with angle \( 2\pi k/n \) and reflections \( sr^k \) (a cross lines through the origin).

In general it can be written as
\[
D_n = \{S^k : 0 \leq k \leq n-1, \theta \leq \phi \leq \pi \}, \text{where } r^n = 1, S^2 = 1, S \cdot r^k \cdot S = r^{-k}.
\]
The cyclic group of order \( n \), \( C_n = \langle r \rangle \) is a normal subgroup of \( D_n \).

Definition (2.12) [9]

The group \( D_{nh} \) is the direct product group \( D_n \times C_2 \), where \( C_2 \) is a cyclic group of order 2 consisting of elements \{1, r'\} with \( (r')^2 = 1 \). It is of order \( 4n \).

Proposition (2.13): [5]

The rational valued characters table of \( D_n \) when \( n \) is an odd number is given as follows:

\[
\equiv^*(D_n) =
\begin{array}{c|c|c}
\Gamma - \text{classes of } C_n & [S] \\
\hline
\theta_1 & \equiv^*(C_n) & 0 \\
\vdots & \vdots & \vdots \\
\theta_{S-1} & 0 \\
\theta_S & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
\theta_{S+1} & 1 & 1 & 1 & \ldots & 1 & 1 & -1 \\
\end{array}
\]

Where \( S \) is the number of \( \Gamma - \) classes of \( C_n \).

Theorem(2.14) : [13]

The rational valued characters table of the group \( D_{nh} \) when \( n \) is an odd number is given as follows:
\[
\equiv^*(D_{nh}) = \equiv^*(D_n) \otimes \equiv^*(C_2)
\]
Theorem (2.15):[2]
Let \( p \) be a prime number, then \( \text{Ar}(C_p^s) = \)

<table>
<thead>
<tr>
<th>( \Gamma )-Classes</th>
<th>[1]</th>
<th>( r^{P_1} )</th>
<th>( r^{P_2} )</th>
<th>( r^{P_3} )</th>
<th>( \ldots )</th>
<th>[r]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_1 )</td>
<td>( p^s )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
</tr>
<tr>
<td>( \varphi_2 )</td>
<td>( p^{s-1} )</td>
<td>( p^{s-1} )</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
</tr>
<tr>
<td>( \varphi_3 )</td>
<td>( p^{s-2} )</td>
<td>( p^{s-2} )</td>
<td>( p^{s-2} )</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( \varphi_s )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \varphi_{s+1} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

Remark (2.16):
Let \( n \) be any positive integer and
\[
\begin{bmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
0 & 1 & 1 & \ldots & 1 & 1 \\
0 & 0 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & -1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & -1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & -1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]

Where \( \otimes \) is the tensor product.

Proposition (2.17):[13]
If \( P \) is a prime number and \( S \) is a positive integer, then
\[
M(C_{p^s}) = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
0 & 1 & 1 & \ldots & 1 & 1 \\
0 & 0 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]
\[
P(C_{p^s}) = \begin{bmatrix}
1 & -1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & -1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & -1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]

And \( W(C_p^s) = I_{S+1} \) where \( I_{S+1} \) is the identity matrix.
Remark (2.18):

1. In general if \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m} \) such that \( p_1, p_2, \cdots, p_m \) are distinct primes and \( \alpha_i \) any positive integers for all \( i = 1, 2, \ldots, m \); then \( C_n = C_{p_1^{\alpha_1}} \times C_{p_2^{\alpha_2}} \times \cdots \times C_{p_m^{\alpha_m}} \). and

\[
M(C_n) = M(C_{p_1^{\alpha_1}}) \otimes M(C_{p_2^{\alpha_2}}) \otimes \cdots \otimes M(C_{p_m^{\alpha_m}}).
\]

So, we can write \( M(C_n) \) as:

\[
M(C_n) = \begin{bmatrix}
1 & & & & \\
& R(C_n) & & & \\
& & 1 & & \\
& & & 1 & \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

Where \( R(C_n) \) is the matrix obtained by omitting the last row \( \{0, 0, \ldots, 0, 1\} \) and the last column \( \{1, 1, \ldots, 1\} \) from the tensor product,

\[
M(C_{p_1^{\alpha_1}}) \otimes M(C_{p_2^{\alpha_2}}) \otimes \cdots \otimes M(C_{p_m^{\alpha_m}}). \quad \text{M(Cn) is}, \quad (\alpha_1 +1)(\alpha_2 +1) \cdots (\alpha_m +1) \times (\alpha_1 +1)(\alpha_2 +1) \cdots (\alpha_m +1) \text{ square matrix.}
\]

2.

\[
\alpha - P(C_n) = P(C_{p_1^{\alpha_1}}) \otimes P(C_{p_2^{\alpha_2}}) \otimes \cdots \otimes P(C_{p_m^{\alpha_m}}).
\]

\[
\beta - W(C_n) = W(C_{p_1^{\alpha_1}}) \otimes W(C_{p_2^{\alpha_2}}) \otimes \cdots \otimes W(C_{p_m^{\alpha_m}}).
\]

3. The Main Results

In this section we give the general form of Artin characters table of the group \( D_{4n} \) and the cyclic decomposition of the factor group \( AC(D_{4n}) \) when \( n \) is an odd number.
Theorem (3.1):
The Artin characters table of the group $D_{n\text{h}}$ when $n$ is an odd number is given as follows:

$$\text{Ar}(D_{n\text{h}}) = \begin{bmatrix}
1 & 1 & 2 & 2 & \ldots & 2 & n & n \\
4n & 4n & 2n & 2n & \ldots & 2n & 4 & 4
\end{bmatrix}$$

Table (3.1)

<table>
<thead>
<tr>
<th>$\Gamma$-Classes</th>
<th>$[1,1']$</th>
<th>$[1,r']$</th>
<th>$\Gamma$-Classes of $C_n \times C_2$</th>
<th>$[S,1]$</th>
<th>$[S,r']$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\text{CL}_\alpha</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$</td>
<td>C_{D_{n\text{h}}}(\text{CL}_\alpha)</td>
<td>$</td>
<td>4n</td>
<td>4n</td>
<td>2n</td>
</tr>
<tr>
<td>$\Phi_{(l,1)}$</td>
<td></td>
<td></td>
<td>2Ar($C_\alpha$) $\otimes$ Ar($C_2$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_{(l,2)}$</td>
<td></td>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\Phi_{(l+1,1)}$</td>
<td></td>
<td></td>
<td>$\Phi_{(l+1,1)}$</td>
<td>2n</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_{(l+1,2)}$</td>
<td></td>
<td></td>
<td>$\Phi_{(l+1,2)}$</td>
<td>2n</td>
<td>0</td>
</tr>
</tbody>
</table>

where $l$ is the number of $\Gamma$-classes of $C_\alpha$ and $C_2 = <r'> = \{1', r'\}$.

Proof:-

By theorem (2.15)

$$\text{Ar}(C_2) = \begin{bmatrix}
\Gamma\text{- classes} & [1'] & [r'] \\
|\text{CL}_\alpha| & 1 & 1 \\
|C_2(\text{CL}_\alpha)| & 2 & 2 \\
\varphi_1' & 2 & 0 \\
\varphi_2' & 1 & 1
\end{bmatrix}$$

Table (3.2)

Each cyclic subgroup of the group $D_{n\text{h}}$ is either a cyclic subgroup of $C_n \times C_2$ or $<(S,r')>$ or $<(S,1')>$.

If H is a cyclic subgroup of $C_n \times C_2$, then:
H=H_1<1'> or H_2< r'> = H_1C_2 for all 1 \leq i \leq l where l is the number of \Gamma- classes of C_n
If H= H_1<1'> and g \in D_{nh}
If g \notin H then by theorem (1.8)
\Phi_{(1,i)}(g)=0  for all 0 \leq i \leq l  [since H \cap CL(g) = \phi]
If g \in H then either g=(1,1') or \exists S, 0 < S < n such that g=(r^S,1')

When g=(1,1') then :
\Phi_{(1,1)}(g) = \left| \frac{C_{D_{nh}}(g)}{C_H(g)} \right| \phi(g) [since H \cap CL(g)=\{(1,1')\}] where \phi is the principle character (i.e
\phi(g)=1 \forall g \in D_{nh})
= \frac{4n}{|H_1 \times <1'>|} \cdot 1 = \frac{4n}{|H_1|} \cdot 1 = 2 \cdot \frac{|C_{C_2}(1)|}{|C_H(1)|} \cdot \phi_1(1) \cdot \phi_1'(1') = 2 \cdot \phi_1(1) \cdot \phi_1'(1')

For g=(r^S,1') then
\Phi_{(i,1)}(g) = \left| \frac{C_{D_{nh}}(g)}{C_H(g)} \right| \sum_{i=1}^{2} \phi'(g) [since H \cap CL(g)=\{(r^S,1'),(r^{-S},1')\}]
= \frac{2n}{|H_1 \times <1'>|} \cdot (1+1)
= \frac{2n}{|H_1|} \cdot 2
= 2 \cdot \frac{n}{|H_1|} \cdot 2 \cdot \frac{|C_{C_2}(r^S)}{|C_H(r^S)|} \cdot \phi(r^S) \cdot \phi'(1') = 2 \cdot \phi_1(r^S) \cdot \phi_1'(1')

If H= H_2< r'> = H_1 \times C_2
let g \in D_{nh}
if g \notin H then
\Phi_{(i,2)}(g)=0  for all 1 \leq i \leq l  [since H \cap CL(g) = \phi]
If g \in H then either g=(1,1') or g=(1,r') or \exists S, 0 < S < n such that g=(r^S,r')

When g=(1,1')
\[ \Phi_{(i,2)}(g) = \frac{|C_{D_{o}}(g)|}{|C_{H}(g)|} \varphi(g) \quad [\text{since } H \cap CL(g) = \{(1,1')\}] \]

\[ = \frac{4n}{|H_i \times C_2|} = \frac{4n}{2|H_i|} = \frac{2n}{|H_i|} = 2 \frac{|C_{o}(1)|}{|C_{H_i}(1)|} \varphi(1) = 2 \cdot \varphi_1(1) \cdot \varphi_2'(1) \]

For \( g = (1,r') \)

\[ \Phi_{(i,2)}(g) = \frac{|C_{D_{o}}(g)|}{|C_{H}(g)|} \varphi(g) \quad [\text{since } H \cap CL(g) = \{(1,r')\}] \]

\[ = \frac{4n}{|H_i \times C_2|} = \frac{4n}{2|H_i|} = 2 \frac{|C_{o}(1)|}{|C_{H_i}(1)|} \varphi(1) = 2 \cdot \varphi_1(1) \cdot \varphi_2'(r') \]

and if \( g = (r^s,r') \)

\[ \Phi_{(i,2)}(g) = \frac{|C_{D_{o}}(g)|}{|C_{H}(g)|} \sum_{i=1}^{2} \varphi'(g) \quad [\text{since } H \cap CL(g) = \{(r^s,r'),(r^{-s},r')\}] \]

\[ = \frac{2n}{|H_i \times C_2|} (1+1) = \frac{4n}{2|H_i|} = \frac{2n}{|H_i|} = 2 \frac{|C_{o}(r')|}{|C_{H_i}(r')|} \varphi(r') \varphi_2'(r') = 2 \cdot \varphi_1(r') \varphi_2'(r'). \]

If \( H = (S,1') = \{(1,1'),(S,1')\} \)

\[ \Phi_{(l+1,1)} ((1,1')) = \frac{|C_{D_{o}}((1,1'))|}{|C_{H}((1,1'))|} \varphi(g) = \frac{4n}{2} = 2n \]

\[ \Phi_{(l+1,1)} ((S,1')) = \frac{|C_{D_{o}}((S,1'))|}{|C_{H}((S,1'))|} \varphi(g) \quad [\text{since } H \cap CL((S,1')) = \{(S,1')\}] \]

\[ = \frac{4}{2} = 2 \]

otherwise

\[ \Phi_{(l+1,1)} (g) = 0 \quad \text{for all } g \in D_{nh}, \text{since } g \notin H \]

If \( H = (S,r') = \{(1,1'),(S,r')\} \)

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\[ \Phi_{(l+1,2)} ((1,1')) = \frac{\begin{bmatrix} C_{D_{\alpha}} ((1,1')) \\ C_{H} ((1,1')) \end{bmatrix} \varphi ((1,1')) }{[since \ H \cap CL((1,1')) = \{ (1,1') \}]} \]

\[ = \frac{4n}{2} \cdot 1 = 2n \]

\[ \Phi_{(l+1,2)} ((S, r')) = \frac{\begin{bmatrix} C_{D_{\alpha}} ((S, r')) \\ C_{H} ((S, r')) \end{bmatrix} \varphi ((S, r')) }{[since \ H \cap CL(g) = \phi]} \]

\[ = 4 \cdot 1 = 2 \]

Otherwise \( \Phi_{(l+1,2)} (g) = 0 \) for all \( g \in D_{nh} \) since \( H \cap CL(g) = \phi \)

**Example (3.2):**

To find \( Ar(D_{2197}) \) by using theorem(3.1)

\( Ar(D_{2197}) = Ar(D_{3^3}) = 2Ar(C_{3^3}) \otimes Ar(C_2) \)

by using theorem(2.15) we get

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|}
\hline
\multicolumn{5}{|c|}{\Gamma-classes} \\
\hline
\( |CL_{\alpha}| \) & \( [1] \) & \( [r^{13}] \) & \( [r^{13}] \) & \( [r] \) \\
\hline
\hline
\( |C_{3^3} (CL_{\alpha})| \) & \( 13^3 \) & \( 13^3 \) & \( 13^3 \) & \( 13^3 \) \\
\hline
\hline
\( \delta_1 \) & \( 13^3 \) & 0 & 0 & 0 \\
\hline
\( \delta_2 \) & \( 13^2 \) & \( 13^2 \) & 0 & 0 \\
\hline
\( \delta_3 \) & 13 & 13 & \( r^{13} \) & 0 \\
\hline
\( \delta_4 \) & 1 & 1 & 1 & 1 \\
\hline
\end{tabular}
\end{table}

\( Table (3.3) \)

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
\multicolumn{3}{|c|}{\Gamma-classes} \\
\hline
\( |CL_{\alpha}| \) & \( [1'] \) & \( [r'] \) \\
\hline
\hline
\( |C_{2} (CL_{\alpha})| \) & 2 & 2 \\
\hline
\hline
\( \delta_1 \) & 2 & 0 \\
\hline
\( \delta^2 \) & 1 & 1 \\
\hline
\end{tabular}
\end{table}

\( Table (3.4) \)
So we get $\text{Ar}(D_{2197h})$

$$\Gamma$$-classes

| $|CL_{\alpha}|$ | $[1,1']$ | $[1,r']$ | $[r_{13}^2,1']$ | $[r_{13}^2,r']$ | $[r_{13},1']$ | $[r_{13},r']$ | $[r,1']$ | $[r,r']$ | $[S,1']$ | $[S,r']$ |
|----------------|----------|----------|-----------------|-----------------|-----------------|-----------------|----------|----------|----------|----------|
| $|C_{C3}(CL_{\alpha})|$ | 8788 | 8788 | 4394 | 4394 | 4394 | 4394 | 4394 | 2197 | 2197 |
| $\phi_{(1,1)}$ | 8788 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\phi_{(1,2)}$ | 4394 | 4394 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\phi_{(2,1)}$ | 676 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\phi_{(2,2)}$ | 338 | 338 | 338 | 338 | 0 | 0 | 0 | 0 | 0 |
| $\phi_{(3,1)}$ | 52 | 0 | 52 | 0 | 52 | 0 | 0 | 0 | 0 |
| $\phi_{(3,2)}$ | 26 | 26 | 26 | 26 | 26 | 0 | 0 | 0 | 0 |
| $\phi_{(4,1)}$ | 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 0 |
| $\phi_{(4,2)}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 0 | 0 |
| $\phi_{(5,1)}$ | 4394 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| $\phi_{(5,2)}$ | 4394 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |

Table(3.5)
Proposition (3.3):

If \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m} \) where \( p_1, p_2, \ldots, p_m \) are distinct primes and \( p_i \neq 2 \) for all \( 1 \leq i \leq m \) and \( \alpha_i \) any positive integers, then:

\[
M(D_{\text{nh}}) = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
2R(C_n) \times M(C_2)
\]

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & \cdots & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

which is \( 2\left[ (\alpha_i + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_m + 1) + 1 \right] \times 2\left[ (\alpha_i + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_m + 1) + 1 \right] \) square matrix.

Proof:

By theorem (3.1) we obtain the Artin characters table \( \text{Ar}(D_{\text{nh}}) \) and from theorem (2.14) we find the rational valued characters table \( \equiv (D_{\text{nh}}) \). Thus by the definition of \( M(G) \) we can find the matrix \( M(D_{\text{nh}}) \) :
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\[
M(D_{nh}) = \text{Ar}(D_{nh}) \cdot (D_{nh})^{-1} = 2R(C_{r}) \times M(C_{z})
\]

Which is \(2\left[(\alpha_{1} + 1) \cdot (\alpha_{2} + 1) \cdots + 1\right] \times 2\left[(\alpha_{1} + 1) \cdot (\alpha_{2} + 1) \cdots + 1\right]\) square matrix.

**Example (3,4):**

To find \(M(D_{1331h}) = M(D_{113h})\) we must find

\[
R(C_{11}^{3}) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

Which is \(3 \times 3\) square matrix.

and \(M(C_{2}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\)

Hence
\[ M(D_{11}^3) = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix} \]

\[ 2R(C_{11^3}) \otimes M(C_2) = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix} \]

\[ M(D_{11}^3) = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix} \]

\[ = \begin{bmatrix}
2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
0 & 2 & 0 & 2 & 0 & 2 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1
\end{bmatrix} \]

Which is 10×10 square matrix
Proposition (3.5):

If \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m} \) such that \( p_1, p_2, \ldots, p_m \) are distinct primes different from 2 and \( \alpha_i \) any positive integers, then

\[
P(D_{nh}) = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
P(C_n) \otimes P(C_2) \\
0 & 0 \\
-1 & -1 \\
0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

and

\[
W(D_{nh}) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & -1 & \cdots & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & -1 & 0 & 1 & 0 \\
1 & 1 & \cdots & 1 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Where \( k = 2([\alpha_1 + 1] \cdot [\alpha_2 + 1] \cdot [\alpha_3 + 1] \cdots [\alpha_m + 1] - 1) \times 2([\alpha_1 + 1] \cdot [\alpha_2 + 1] \cdot [\alpha_3 + 1] \cdots [\alpha_m + 1] - 1) \)

They are \( 2([\alpha_1 + 1] \cdot [\alpha_2 + 1] \cdots [\alpha_m + 1] + 1) \times 2([\alpha_1 + 1] \cdot [\alpha_2 + 1] \cdots [\alpha_m + 1] + 1) \) square matrix.

**Proof:**

By using theorem (2.4) and taking the form of \( M(D_{nh}) \) from proposition (3.3) and the above forms of \( P(D_{nh}) \) and \( W(D_{nh}) \) then we

Have

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P(D_{nh}), \ M(D_{nh}), \ W(D_{nh})=
\begin{bmatrix}
2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}

= \text{diag}\{2,2,\ldots,-2,1,1,1\} = D(D_{nh})

Which is $2[(\alpha_1+1)\cdot(\alpha_2+1)\cdot(\alpha_m+1)+1] \times 2[(\alpha_1+1)\cdot(\alpha_2+1)\cdot(\alpha_m+1)+1]$ square matrix.

**Example (3.6):**

Consider the group $D_{27869h}$, then we can find the matrices $P(D_{27869h})$, $W(D_{27869h})$ immediately by using proposition (3.5) and by proposition (3.3) we find $M(D_{27869h})$.

To find $P(D_{27869h}) = P(D_{31^2.29h})$

By remark (2.18-2) $P(C_{31^2.29}) = P(C_{31^2}) \otimes P(C_{29})$

Then $P(C_{31^2}) \otimes P(C_{29}) \otimes P(C_2) = \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \otimes \begin{bmatrix}
1 & -1 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} \otimes \begin{bmatrix}
1 & -1 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}$

$$= \begin{bmatrix}
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

By proposition (3.5)
Then \( P(D_{27869h})\). \( M(D_{27869h})\). \( W(D_{27869h}) = \)

\[
\begin{bmatrix}
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times
\begin{bmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
Which is \(2^{[(2+1)-(1+1)+1]}2^{[(2+1)-(1+1)+1]}=14 \times 14\) square matrix.

**Theorem (3.7):**

If \(n = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdots P_m^{\alpha_m}\) where \(P_1, P_2, \cdots, P_m\) are distinct primes, \(P_i \neq 2\) and \(\alpha_i\) any positive integers for all \(i, 1 \leq i \leq m\), then the cyclic decomposition \(AC(D_{nh})\) is:
\[
2((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1))-1
\]

\[
\text{AC}(D_{\text{nh}}) = \bigoplus_{i=1}^{2} C_2
\]

\[
= \bigoplus_{i=1}^{2} \text{AC}(D_n) \bigoplus C_2
\]

**Proof:**

From proposition (3.5) we have

\[
P(D_{\text{nh}}) \cdot M(D_{\text{nh}}) \cdot W(D_{\text{nh}}) = \text{diag}\{2,2,2,\ldots,-2,1,1,1\} = \{d_1,d_2,\ldots\},
\]

\[
d_{2((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1))-1},
\]

\[
d_{2((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1))-1}.
\]

By theorem (2.7) we get

\[
2((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1))-1
\]

\[
\text{AC}(D_{\text{nh}}) = \bigoplus_{i=1}^{d_i} C_2
\]

\[
= \bigoplus_{i=1}^{2} \text{AC}(D_n) \bigoplus C_2
\]

From theorem (2.8) we have:

\[
\text{AC}(D_{\text{nh}}) = \bigoplus_{i=1}^{2} \text{AC}(D_n) \bigoplus C_2
\]

**Example (3.8):**

To find the cyclic decomposition of the groups AC(D_{29791h}), AC(D_{25054231h})

and AC(D_{576247313h}).

by using above theorem:

\[
2(3+1)-1 = 7
\]

\[
\text{AC}(D_{29791h}) = \bigoplus_{i=1}^{2} C_2 = \bigoplus_{i=1}^{7} C_2 = \bigoplus_{i=1}^{2} \text{AC}(D_{31^3}) \bigoplus C_2.
\]
\[
2((3+1)(2+1))-1 = 23 \\
AC(D_{2504231h}) = AC(D_{31^{\frac{3}{2}2^{\frac{1}{2}}}}) = \bigoplus_{i=1}^{2} C_2 = \bigoplus_{i=1}^{2} C_2 \\
\]

\[
2 = \bigoplus_{i=1}^{2} AC(D_{31^{\frac{3}{2}2^{\frac{1}{2}}}} C_2 . \\
2((3+1)(2+1)(1+1))-1 = 47 \\
2 \cdot AC(D_{576247313h}) = AC(D_{31^{\frac{3}{2}2^{\frac{1}{2}}2^{\frac{1}{2}}}}) = \bigoplus_{i=1}^{2} C_2 = \bigoplus_{i=1}^{2} C_2 \\
\]

\[
2 = \bigoplus_{i=1}^{2} AC(D_{31^{\frac{3}{2}2^{\frac{1}{2}}2^{\frac{1}{2}}}} C_2 . \\
\]

References :-