

δ^* -Base for a tritopology

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Abstract :-

In this work , we introduce a new definitions for δ^* -Base for the δ^* -Neighbourhood system of a point (δ^* - Local Base) , δ^* -Base for a tritopology and δ^* -sub Base for a tritopology . And we gives the basic specifications for the new definitions .

Keywords :-

δ^* - Local Base , δ^* - Base , δ^* -sub Base .

الخلاصة :

في هذا العمل قدمت تعاريف جديدة في الفضاء الثلاثي التبولوجي و هي أساس δ^* - لنظام الجوارات لنقطة (أساس محلي δ^* -) و أساس δ^* - للفضاء الثلاثي التبولوجي و أساس δ^* - جزئي للفضاء الثلاثي التبولوجي . و كما قمت بإعطاء المواصفات الأساسية لهذه التعاريف في الفضاء الثلاثي التبولوجي .

Introduction :-

Throughout this paper we adopt the notations and terminology of [1] , [2] , [3] and [4] , X and Y are finite sets and the following conventions : (X,T) , (X,T,Ω) , (X,T,Ω,ρ) will always denot to Topological space , Bitopological space and Tritopological space respectively .

Let (X,T) be a topological space , and let A be a subset of X , then A is said to be a α -open set iff $A \subseteq T - \text{int}(T - \text{cl}(T - \text{int}(A)))$ [1] , and the family of all α -open sets is denoted by $\alpha.O(X)$. The complement of α -open set is called α -closed set .

Let (X,T,Ω) be a Bitopological space , and let A be a subset of X , then A is said to be a δ -open set iff $A \subseteq T - \text{int}(\Omega - \text{cl}(T - \text{int}(A)))$ [3] , and the family of all δ -open sets is denoted by $\delta.O(X)$. The complement of δ -open set is called δ -closed set .

And let (X,T,Ω,ρ) be a Tritopological space , a subset A of X is said to be δ^* -open set iff $A \subseteq T - \text{int}(\Omega - \text{cl}(\rho - \text{int}(A)))$ [2] , and the family of all δ^* -open sets is denoted by $\delta^*.O(X)$. The complement of δ^* -open set is called a δ^* -closed set .

And $\delta^*.O(X)$ does not represent a topology (not always represent a topology) [2] , (X,T,Ω,ρ) is called discrete tritopological space with respect to δ^* -open if $\delta^*.O(X)$ contains all subsets on X .

(X,T,Ω,ρ) is called indiscrete topological space with respect to δ^* -open if $\delta^*.O(X) = \{ X , \varphi \}$.

Let (X,T,Ω,ρ) be a tritopological space , and let $x \in X$. A subset N of X is said to be δ^* -nhd of a point x iff there exists δ^* -open set U such that $x \in U \subset N$. The set of all δ^* -nhds of a point x is denoted by $\delta^* - N(x)$.

1- δ^* - Local Base at a point :

1.1 Definition :

Let (X,T,Ω,ρ) be a tritopological space . A non empty collection $\delta^*-\beta(x)$ of δ^* -neighbourhoods of x is called a δ^* -base for the δ^* -neighbourhood system of x (δ^* - Local Base) iff for every δ^* -neighbourhood N of x there is a $B \in \delta^*-\beta(x)$ such that $B \subset N$.

We then also say that $\delta^*-\beta(x)$ is a δ^* - Local Base at x or a fundamental system of δ^* -neighbourhoods of x .

If $\delta^*-\beta(x)$ is a δ^* - Local Base at x , then the members of $\delta^*-\beta(x)$ are called basic δ^* -neighbourhoods of x .

1.2 Remark :

If $\delta^*.O(X)$ represent a topology then every point x in X has a δ^* - local base in tritopology .

Proof :

Clearly because the intersection condition of the topology definition is hold .

1.3 Example :

$$\begin{aligned} \text{Let } X = \{a,b,c,d\} & , & T = \{ X , \varphi , \{b,c,d\} \} . \\ & , & \Omega = \{ X , \varphi , \{a\} \} \\ & & \rho = \{ X, \varphi, \{a\}, \{c,d\}, \{c\}, \{b,c\}, \{a,c,d\}, \\ & & \{a,c\}, \{a,b,c\}, \{b,c,d\} \} \end{aligned}$$

(X,T) , (X,Ω) , (X,ρ) are three topological spaces , then (X,T,Ω,ρ) is a tritopological space , such that :

$$\delta^*.O(X) = \{ X, \varphi, \{a\}, \{c,d\}, \{c\}, \{b,c\}, \{a,c,d\}, \{a,b,c\}, \{a,c\}, \{a,b\}, \{a,d\}, \{a,b,d\}, \{b,c,d\} \}$$

Clearly that $\delta^*.O(X)$ represent a topology . Then the δ^* - local base at each of the points a , b , c , d is given by $\delta^*-\beta(a) = \{ \{a\} \}$, $\delta^*-\beta(b) = \{ \{b,c\} \}$, $\delta^*-\beta(c) = \{ \{c\} \}$, $\delta^*-\beta(d) = \{ \{c,d\} \}$.

Observe that here a δ^* - Local Base at each point consists of a single δ^* -nhd of the point .

1.4 Remark :

If $\delta^*.O(X)$ does not represent a topology then not every point x in X has a δ^* - local base in tritopology .

Proof :

Clearly because the intersection condition of the topology definition is not hold .

1.5 Example :

Let $X=\{a,b,c,d\}$, $T = \{X, \varphi, \{c,d\}\}$
 $\Omega = \{X, \varphi, \{a,b,c\}, \{a\}\}$
 and $\rho = \{X, \varphi, \{d\}, \{c,d\}, \{a,d\}\}$

(X,T) , (X,Ω) , (X,ρ) are three topological spaces , then (X,T,Ω,ρ) is a tritopological space , such that :

$$\delta^*.O(X) = \{X, \varphi, \{a,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$$

Clearly that $\delta^*.O(X)$ does not represent a topology . Then the δ^* - Local Base at each of the points a , b , c , d is given by :

$$\delta^*-\beta(a) = \{ \{a,d\} \} , \delta^*-\beta(c) = \{ \{c,d\} \}$$

But there is no $\delta^*-\beta(b)$ because the intersection of $\{a,b,d\}$ and $\{b,c,d\}$ is $\{b,d\}$ is not δ^* - open set (not exist) , i.e. there is no set B such that B is a subset of all δ^* -nhds of b .

Again , there is no $\delta^*-\beta(d)$ because the intersection of $\{a,d\}$ and $\{c,d\}$ is $\{d\}$ is not δ^* - open set (not exist) , i.e. there is no set D such that D is a subset of all δ^* -nhds of d .

Hence not every point x in X has a δ^* - local base in tritopology .

1.6 Remark :

All the following theorems and remarks is true in tritopology if $\delta^*.O(X)$ represent a topology (i.e. if every x has δ^* - local base) .

Proof:

It is clear by the remarks and examples above .

1.7 Theorem :

Let (X,T,Ω,ρ) be a tritopological space . And let $x \in X$ then the collection $\delta^*-\beta(x)$ of all δ^* - open subsets of X containing x is a δ^* - Local Base at x .

Proof :

Let N be any δ^* -nhd of x . Then there exists a δ^* -open set G such that $x \in G \subset N$, since G is a δ^* -open set containing x , $G \in \delta^*-\beta(x)$.

This shows that $\delta^*-\beta(x)$ is a δ^* - Local Base at x .

1.8 Remark :

(I) In a discrete tritopological space the collection $\{ \{x\} \}$ consisting of a single member $\{x\}$ forms a δ^* - Local Base at $x \in X$

Proof :

for $\{x\}$ is a δ^* -open subset of X and so it is a δ^* -nhd of x and any other δ^* -nhd N of x must contain $\{x\}$.

(II) In a indiscrete tritopological space , the set X alone forms a δ^* - Local Base for every point $x \in X$.

Proof :

There is only δ^* -open X and \varnothing in $\delta^*.O(X)$. and must be X .

1.9 Theorem :

Let (X, T, Ω, ρ) be a tritopological space . And let $\delta^*-\beta(x)$ be a δ^* - Local Base for every point $x \in X$. Then $\delta^*-\beta(x)$ has the following properties :

- (1) $\delta^*-\beta(x) \neq \varnothing$ for every $x \in X$.
- (2) If $B \in \delta^*-\beta(x)$, then $x \in B$.
- (3) If $A \in \delta^*-\beta(x)$ and $B \in \delta^*-\beta(x)$, then there exists a set $C \in \delta^*-\beta(x)$ such that $C \subset A \cap B$.
- (4) If $A \in \delta^*-\beta(x)$, then there exists a set B such that $x \in B \subset A$ and such that for every point $y \in B$, there exists $C \in \delta^*-\beta(y)$ satisfying $C \subset B$.

Proof :

(1) Since X is δ^* -open , it is δ^* -nhd of its point . Since $\delta^*-\beta(x)$ is δ^* - Local Base for every point $x \in X$ and X is a δ^* -nhd of x , it follows that there must exist a $B \in \delta^*-\beta(x)$ such that $B \subset X$.

Hence $\delta^*-\beta(x) \neq \varnothing$ for every $x \in X$.

(2) If $B \in \delta^*-\beta(x)$, then B is a δ^* -nhd of x . So by definition of δ^* -nhd $x \in B$.

(3) If $A \in \delta^*-\beta(x)$, then A is a δ^* -nhd of x . Similarly B is a δ^* -nhd of x . It follows that $A \cap B$ is δ^* -nhd of x . Since $\delta^*-\beta(x)$ is δ^* - Local Base at x , it follows that there exists $C \in \delta^*-\beta(x)$ such that $C \subset A \cap B$.

(4) Since $A \in \delta^*-\beta(x)$, A is a δ^* -nhd of x . Hence there exists a δ^* -open set B such that $x \in B \subset A$. Since B is δ^* -open set , it is a δ^* -nhd for every $y \in B$. Again since $\delta^*-\beta(y)$ is δ^* - Local Base at y and B is a δ^* -nhd for every $y \in B$, it follows that for every $y \in B$ there exists $C \in \delta^*-\beta(y)$ such that $C \subset B$.

2- δ^* - Base for a tritopology :

2.1 Definition :

Let (X, T, Ω, ρ) be a tritopological space . A collection $\delta^*-\beta$ of subsets of X is said to form a δ^* - base for the tritopology (T, Ω, ρ) iff :

(I) $\delta^*-\beta \subset \delta^*.O(X)$.

(II) for each point $x \in X$ and each δ^* -neighbourhood N of x there exists some $B \in \delta^*-\beta$ such that $x \in B \subset N$.

2.2 Remark :

If $\delta^*.O(X)$ represent a topology then the tritopology (T, Ω, ρ) has a δ^* - base .

Proof :

Clearly because the intersection condition of the topology definition is hold .

2.3 Example :

Let $X = \{a,b,c\}$, $T = \{ X , \varphi , \{a,b\} \}$.
 $\Omega = \{ X , \varphi , \{c\} \}$
 and $\rho = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^* . O(X) = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$$

Clearly that $\delta^* . O(X)$ represent a topology .

Then the collection $\delta^* - \beta = \{X, \{a\}, \{b\}\}$ is a δ^* - base for the tritopology (T, Ω, ρ) since :

- (I) $\delta^* - \beta \subset \delta^* . O(X)$. And
- (II) Each δ^* -neighbourhood of a contains $\{a\}$ which is a member of $\delta^* - \beta$ containing a . Similarly each of δ^* -neighbourhood of b contains $\{b\}$ which is a member of $\delta^* - \beta$ containing b . And X is a δ^* -neighbourhood of c which is a member of $\delta^* - \beta$ containing c .

2.4 Remark :

If $\delta^* . O(X)$ does not represent a topology then the tritopology (T, Ω, ρ) has not a δ^* - base .

Proof :

Clearly because the intersection condition of the topology definition is not hold .

2.5 Example :

Let $X = \{a,b,c,d\}$, $T = \{X, \varphi, \{c,d\}\}$
 $\Omega = \{X, \varphi, \{a,b,c\}, \{a\}\}$
 and $\rho = \{X, \varphi, \{d\}, \{c,d\}, \{a,d\}\}$

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^* . O(X) = \{X, \varphi, \{a,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$$

Clearly that $\delta^* . O(X)$ does not represent a topology .

Then this tritopology has not a δ^* - base because the second condition of the definition (2.1) is not hold . because the intersection of $\{a,b,d\}$ and $\{b,c,d\}$ is $\{b,d\}$ is not a δ^* - open set (not exist) , i.e. each δ^* -nhds of b contains $\{b,d\}$, but $\{b,d\}$ is not exist in $\delta^* . O(X)$ because $\delta^* . O(X)$ does not represent a topology . .

Again , because the intersection of $\{a,d\}$ and $\{c,d\}$ is $\{d\}$ is not a δ^* - open set (not exist) , i.e. each δ^* -nhds of d contains $\{d\}$, but $\{d\}$ is not exist in $\delta^* . O(X)$ because $\delta^* . O(X)$ does not represent a topology .

Hence there is no a δ^* - base .

2.6 Remark :

In the discrete tritopological space the collection $\delta^*-\beta = \{ \{x\} : x \in X \}$ consisting of all singleton subsets of X is a δ^* -Base for the tritopology (T, Ω, ρ) .

Proof:

Because for each singleton set is δ^* -open so that $\delta^*-\beta \subset \delta^*.O(X)$. Also for each $x \in X$ and each δ^* -nhds N of x , $\{x\} \in \delta^*-\beta$ is such that $x \in \{x\} \subset N$.

2.7 Remark :

All the following theorems and remarks is true in tritopology if $\delta^*.O(X)$ represent a topology (i.e. if (T, Ω, ρ) has δ^* -base).

Proof :

It is clear by the remarks (2.2) and (2.4) and examples (2.3) and (2.5) .

2.8 Theorem :

Let (X, T, Ω, ρ) be a tritopological space . A sub collection $\delta^*-\beta$ of $\delta^*.O(X)$ is a δ^* -Base for (T, Ω, ρ) iff every δ^* -open set can be expressed as the union of members of $\delta^*-\beta$.

Proof :

Let $\delta^*-\beta$ is a δ^* -Base for (T, Ω, ρ) and let $G \in \delta^*.O(X)$. Since G is δ^* -open set , it is a δ^* -nhd of each of its points . Hence by definition of δ^* -Base , to each $x \in G \exists$ a member B of $\delta^*-\beta$ s.t.

$$x \in B \subset G$$

It follows that $G = \cup \{ B : B \in \delta^*-\beta \text{ and } B \subset G \}$.

Conversely , let $\delta^*-\beta \subset \delta^*.O(X)$ and let every δ^* -open set G be the union of members of $\delta^*-\beta$.

We have to show that $\delta^*-\beta$ is δ^* -Base for (T, Ω, ρ) . We have

- (1) $\delta^*-\beta \subset \delta^*.O(X)$ (given)
- (2) let $x \in X$ and let N be any δ^* -nhd of x . Then there exists a δ^* -open set G such that $x \in G \subset N$.

But G is the union of members of $\delta^*-\beta$. Hence there exists $B \in \delta^*-\beta$ such that $x \in B \subset G \subset N$. Thus $\delta^*-\beta$ is δ^* -Base for (T, Ω, ρ) .

2.9 Theorem :

Let (T_1, Ω_1, ρ_1) and (T_2, Ω_2, ρ_2) be a tritopologies for X which have a common δ^* -Base $\delta^*-\beta$. Then $\delta^*_1.O(X) = \delta^*_2.O(X)$.

Proof :

Let $G \in \delta^*_1.O(X)$ and $x \in G$. Since G is δ^*_1 -open set , it is a δ^*_1 -nhd of x and since $\delta^*-\beta$ is a δ^* -Base for (T_1, Ω_1, ρ_1) , there exists $B \in \delta^*-\beta$ such that $x \in B \subset G$. Since $\delta^*-\beta$ is a δ^* -Base for (T_2, Ω_2, ρ_2) , and $B \in \delta^*-\beta$ it follows that $B \in \delta^*_2.O(X)$. Hence G is δ^*_1 -nhd

of x . Since x is arbitrary .Then $G \in \delta^*_2.O(X)$ by theorem [$G \in \delta^*.O(X)$ iff G is a δ^* - nhd of each of its points] . Thus $\delta^*_1.O(X) \subset \delta^*_2.O(X)$.By symmetry $\delta^*_2.O(X) \subset \delta^*_1.O(X)$. Hence $\delta^*_1.O(X) = \delta^*_2.O(X)$.

● **Properties of a δ^* - Base for a tritopology .**

2.10 Theorem :

Let (X,T,Ω,ρ) be a tritopological space . And Let $\delta^*-\beta$ is a δ^* - Base for (T,Ω,ρ) . Then $\delta^*-\beta$ has the following properties :

- (1) for every $x \in X$, there exists a $B \in \delta^*-\beta$ such that $x \in B$, that is , $X = \cup \{B : B \in \delta^*-\beta\}$
- (2) for every $B_1 \in \delta^*-\beta$, $B_2 \in \delta^*-\beta$ and every point $x \in B_1 \cap B_2$, there exists a $B \in \delta^*-\beta$ such that $x \in B \subset B_1 \cap B_2$, that is , the intersection of any two members of $\delta^*-\beta$ is a union of members of $\delta^*-\beta$.

Proof :

- (1) Since X is a δ^* -open set , it is a δ^* - nhd of each of its points . Hence by definition of δ^* - Base , for every $x \in X$, there exists some $B \in \delta^*-\beta$ such that $x \in B \subset X$ in other words , $X = \cup \{B : B \in \delta^*-\beta\}$.
- (2) If $B_1 \in \delta^*-\beta$ and $B_2 \in \delta^*-\beta$, then B_1 and B_2 are δ^* -open sets . Hence their intersection $B_1 \cap B_2$ is also δ^* -open and therefore $B_1 \cap B_2$ is a δ^* - nhd of each of its points . So by definition of δ^* - Base , to each $x \in B_1 \cap B_2$, there exists $B \in \delta^*-\beta$ such that $x \in B \subset B_1 \cap B_2$, that is , $B_1 \cap B_2$ is the union of members of $\delta^*-\beta$.

2.10 Theorem :

Let (X,T,Ω,ρ) be a discrete tritopological space and let $\delta^*-\beta$ be the collection of all singleton subsets of X , any class $\delta^*-\beta'$ of subsets of X is a δ^* - Base for the discrete tritopology (T,Ω,ρ) iff $\delta^*-\beta \subset \delta^*-\beta'$.

Proof :

Let $\delta^*-\beta'$ be a δ^* - Base for the discrete tritopology (T,Ω,ρ) . Since each singleton $\{x\}$ is δ^* - open set , $\{x\}$ must be a union of members of $\delta^*-\beta'$. But a singleton set can only be the union of itself or itself with the empty set \emptyset . Hence $\{x\}$ must belong to $\delta^*-\beta'$. It follows that $\delta^*-\beta \subset \delta^*-\beta'$.

Conversely , let $\delta^*-\beta \subset \delta^*-\beta'$. Now $\delta^*-\beta = \{ \{x\} : x \in X \}$ is a δ^* - Base for the discrete tritopology (T,Ω,ρ) by remark (2.6) . Also each member of $\delta^*-\beta'$ is δ^* -open so that $\delta^*-\beta' \subset \delta^*.O(X)$.

Since $\delta^*-\beta \subset \delta^*-\beta'$ it follows that if any δ^* -open set G is the union of members of $\delta^*-\beta$, it is also the union of members of $\delta^*-\beta'$. Hence $\delta^*-\beta'$ is also a δ^* - Base for the discrete tritopology (T,Ω,ρ) .

3- δ^* - Sub base for a tritopology :

3.1 Definition :

Let (X, T, Ω, ρ) be a tritopological space . A collection $\delta^* - \beta_*$ of subsets of X is called a δ^* - sub base for the tritopology (T, Ω, ρ) iff : $\delta^* - \beta_* \subset \delta^* . O(X)$. And finite intersections of members of $\delta^* - \beta_*$ form a δ^* - base for the tritopology (T, Ω, ρ) .

It follows that $\delta^* - \beta_*$ is a δ^* - sub base for the tritopology (T, Ω, ρ) iff every member of $\delta^* . O(X)$ is the union of finite intersections of members of $\delta^* - \beta_*$

3.2 Remark :

If $\delta^* . O(X)$ represent a topology then the tritopology (T, Ω, ρ) has a δ^* - sub base .
Proof :

Clearly by using remark (2.2) .

3.3 Example :

Let $X = \{a, b, c, d\}$, $T = \{ X, \varnothing, \{a, c, d\} \}$.
 $\Omega = \{ X, \varnothing, \{b\} \}$
 and $\rho = \{ X, \varnothing, \{a\} \}$

(X, T) , (X, Ω) , (X, ρ) are three topological spaces , then (X, T, Ω, ρ) is a tritopological space , such that :

$$\delta^* . O(X) = \{ X, \varnothing, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\} \}$$

Clearly that $\delta^* . O(X)$ represent a topology .

Then the collection $\delta^* - \beta_* = \{ \{a, c\}, \{a, d\} \}$ is a δ^* - sub base for the tritopology (T, Ω, ρ) since the family $\delta^* - \beta$ of finite intersections of $\delta^* - \beta_*$ is given by :

$$\delta^* - \beta = \{ \{a\}, \{a, c\}, \{a, d\} \}$$

Which is a δ^* - base for the tritopology (T, Ω, ρ) .

3.4 Remark :

If $\delta^* . O(X)$ does not represent a topology then the tritopology (T, Ω, ρ) has not a δ^* - sub base .

Proof :

Clearly by using remark (2.4) .

3.5 Example :

Let $X = \{a, b, c, d\}$, $T = \{ X, \varnothing, \{c, d\} \}$
 $\Omega = \{ X, \varnothing, \{a, b, c\}, \{a\} \}$
 and $\rho = \{ X, \varnothing, \{d\}, \{c, d\}, \{a, d\} \}$

$(X,T), (X,\Omega), (X,\rho)$ are three topological spaces , then (X,T,Ω,ρ) is a tritopological space , such that :

$$\delta^* . O(X) = \{X, \varphi, \{a,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$$

Clearly that $\delta^* . O(X)$ does not represent a topology .

Then this tritopology has not a δ^* - sub base because it has not δ^* - base because $\delta^* . O(X)$ does not represent a topology , then the second condition of the definition (2.1) is not hold .

Hence there is no a δ^* - sub base .

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