

On Pre-connected sets in Bitopological Spaces

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Abstract

In this paper, we give and study a generalization for the concept "pre-connected set" in bitopological spaces, through as well as we give several various results related to this generalization.

Introduction

The study of pre-connected sets in Bitopological Space takes place at several sources and researchers like Kelly (1963), Valiru (1977), Jelles (1989), Al-Swidi (1993) and Dontcher (1998). A generalization of the definition of the pre-connected set is presented with some theorems and examples to explain and illustrative it. Conclusions are added at the end of it.

1. Some definitions

Definition (1) : [2]

Let (X, τ) be a topological space. A subset A of X is said to be *pre-open* set iff $A \subseteq \text{int}_{\tau}(cl_{\tau}(A))$. The family of all *pre-open* sets is denoted by $pr-O(X)$.

Definition (2) : [3]

Let (X, τ, ρ) be a bitopological space. A subset A of X is called *pre-open set* with respect to the two topologies τ, ρ if $A \subseteq \text{int}_{\tau}[cl_{\rho}(A)]$.

The collection of all pre-open sets with respect to the two topologies τ and ρ is denoted by $pr-O(X)$.

Definition (3) : [2]

Let (X, τ, ρ) be a bitopological space. A subset A of X is called *pre-closed* set of X iff the complement of A is *pre-open* set of X . The collection of all *pre-closed* sets with respect to the two topologies τ and ρ denoted by $pr-C(X)$.

Note (1) :

The family of all *pre-open* sets of X is not necessary a topology on X , see the following example.

Example (1) :

Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, and $\rho = \{X, \phi, \{c\}\}$. (X, τ) , (X, ρ) are two topological spaces. Then (X, τ, ρ) is bitopological space, so :

$$pr-O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

Definition (4) : [6]

Let (X, τ, ρ) be a bitopological space. Two non- empty subsets A and B of X are said to be pr -separated iff $[A \cap pr-cl(B)] \cup [B \cap pr-cl(A)] = \phi$. That is A and B are pr -separated sets iff A and B are disjoint and every one contains no limit points of the other with respected to $pr-O(X)$.

Example (2) :

Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\rho = \{X, \phi, \{c\}, \{a, b\}\}$. $(X, \tau), (X, \rho)$ are two topological spaces, then (X, τ, ρ) is a bitopological space, so, $pr-O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let $A = \{a\}$ and $B = \{b\}$ are two pr -open sets.

Hence A and B are pr -separated sets, since

$$pr-cl(A) = pr-cl(\{a\}) = \{a\}, pr-cl(B) = pr-cl(\{b\}) = \{b\},$$

$$\text{and } [A \cap pr-cl(B)] \cup [pr-cl(A) \cap B] = [\{a\} \cap \{b\}] \cup [\{a\} \cap \{b\}] = \phi \cup \phi = \phi.$$

So, A and B are pr -separated subsets of X

Definition (5) : [6]

Let (X, τ, ρ) be a bitopological space. A subset A of X is said to be pr -disconnected set iff it is the union of two pr -separated non-empty sets in X , that is \exists two non-empty sets C and D in X such that

$$pr-cl(C) \cap D = \phi, \quad C \cap pr-cl(D) = \phi \text{ and } A = C \cup D.$$

A is called pr -connected iff it is not pr -disconnected set.

Example (3) :

Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and

$\rho = \{X, \phi, \{b\}, \{a, c\}\}$. $(X, \tau), (X, \rho)$ are two topological spaces. Then (X, τ, ρ) is a bitopological space, so, $pr-O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$,

$$\text{Let } A = \{a, b\} \text{ and } C = \{a\}, D = \{b\} \text{ so } A = C \cup D.$$

$$\text{Hence } pr-cl(C) = \{a\}, pr-cl(D) = \{b\},$$

$$[pr-cl(C) \cap D] \cup [C \cap pr-cl(D)] = [\{a\} \cap \{b\}] \cup [\{a\} \cap \{b\}] = \phi.$$

Therefore A is pr -disconnected set, but if $B = \{b, c\}, C = \{b\}, D = \{b, c\}$

$$\text{hence } pr-cl(C) = \{b\}, pr-cl(D) = \{b, c\}$$

$$[pr-cl(C) \cap D] \cup [C \cap pr-cl(D)] = [\{b\} \cap \{b, c\}] \cup [\{b\} \cap \{b, c\}] = \{b\} \cup \{b\} = \{b\} \neq \phi$$

Then B is pr -connected set.

Notes (2) :

1. The empty set is trivially pr -connected.
2. Every singleton pr -open set is pr -connected set, since it can not be expressed as a union of two non-empty pr -separated sets.

Definition (6) : [6]

Two points a, b of a bitopological space (X, τ, ρ) are said to be pr -connected iff they are contained in a pr -connected subset of X .

Definition (7) : [4]

Let (X, τ, ρ) be a bitopological space, and Y be a subset of X . The relative bitopological space for Y is denoted by (Y, τ_Y, ρ_Y) , such that :

$$\tau_Y = \{G \cap Y : G \in \tau\}$$

$$\rho_Y = \{H \cap Y : H \in \rho\}$$

(Y, τ_Y, ρ_Y) is called a subspace of the bitopological space (X, τ, ρ) , the relative bitopological space for Y with respect to pr -open sets is the collection

$$pr-O(X)_Y \quad \text{given by}$$

$$pr-O(X)_Y = \{G \cap Y : G \in pr-O(X)\}$$

2. Various results :

Theorem (1) :

Let (Y, τ_Y, ρ_Y) be subspace of (X, τ, ρ) let $A, B \subset pr-O(X)$ then A and B are $pr-O(X)$ -separated iff they $pr-O(X)_Y$ -separated.

Theorem (2) :

Let (Y, τ_Y, ρ_Y) be a subspace of bitopological (X, τ, ρ) and $A \subset Y$. Then A is pr -disconnected iff it is pr_Y -disconnected .

Proof :

By above theorem, therefore, A is the union of two pr -separated sets iff it is the union of two pr_Y -separated sets .

Theorem (3) :

A bitopological space (X, τ, ρ) is pr -disconnected iff \exists a non-empty proper subset of X which is both pr -open and pr -closed in X .

Proof :

\Leftarrow

Let A a non-empty proper subset of X which both pr -open and pr -closed. We will show that X is a pr -disconnected .

Let $B = A^c \Rightarrow B \neq \emptyset$, since $A \subset X$, so $B \cup A = X$ and $A \cap B = \emptyset$ (*)

Since A is both pr -open and pr -closed $\Rightarrow B$ is both pr -open and pr -closed , hence $pr-cl(A) = A, pr-cl(B) = B$.

Thus X is pr -disconnected .

\Rightarrow

Let X is pr -disconnected ,Then \exists two non-empty pr -separated subsets A and $B \ni X = A \cup B$ (**)
 $pr-cl(A) \cap B = \emptyset$ and
 $A \cap pr-cl(B) = \emptyset$,
 since $A \subset pr-cl(A)$ and
 $pr-cl(A) \cap B = \emptyset \Rightarrow A \cap B = \emptyset$.

Hence , $A = B^c$, by (**)
 $B \neq \emptyset \Rightarrow X \subseteq A \cup pr-cl(B)$ since
 $(B \subset pr-cl(B))$.

But

$A \cup pr-cl(B) \subseteq X \Rightarrow X = A \cup pr-cl(B)$
 , also

$$A \cap pr-cl(B) = \emptyset \Rightarrow A = [pr-cl(B)]^c .$$

Similarity , $B = [pr-cl(A)]^c$. Since $pr-cl(A)$ and $pr-cl(B)$ are pr -closed sets, therefore A and B are pr -open sets, and hence $A = B^c$ is a pr -closed set (since $A \cap B = \emptyset$) . Thus A is a non-empty pr -closed and pr -open set .

Theorem (4) :

Let (X, τ, ρ) be a bitopological space. Then X is a pr -disconnected iff any one of the following statements holds :

- i. X is the union of two non-empty disjoint pr -open sets .
- ii. X is the union of two non-empty disjoint pr -closed sets .

Proof :

\Rightarrow

Let X be a pr -disconnected $\Rightarrow \exists A \neq \emptyset, A \subset X$ which is both pr -closed and pr -open set . Therefore A^c is also both pr -closed and pr -open set and $A \cup A^c = X$.

Hence the sets A and A^c are satisfy (i) and (ii) .

←

Let $X = A \cup B$ and $A \cap B = \phi$, where A and B are non-empty pr -open (pr -closed) sets $\Rightarrow A = B^c$, so that A is pr -closed(pr -open) set.

Since

$B \neq \phi, B \subset X \Rightarrow A \neq \phi, A \subset X$ which is both pr -open and pr -closed .

Hence X is a pr -disconnected .

Theorem (5) :

Let E be a pr -connected subset of a bitopological space (X, τ, ρ) . If $F \subset X$ such that $E \subset F \subset pr-cl(E)$, then F is a pr -connected . In particular , $pr-cl(E)$ is pr -connected .

Proof :

Suppose that F is pr -disconnected

$\Rightarrow \exists A, B \neq \phi, A \cap pr-cl(B) = \phi, pr-cl(A) \cap B = \phi, A \cup B = F$, since

$E \subset F = A \cup B \Rightarrow E \subset A$ or $E \subset B$.

Let $E \subset A \Rightarrow pr-cl(E) \subset pr-cl(A) \Rightarrow pr-cl(E) \cap B \subset pr-cl(A) \cap B = \phi \Rightarrow pr-cl(E) \cap B = \phi$(*)

Also, $A \cup B = F \subset pr-cl(E) \Rightarrow B \subset F \subset pr-cl(E) \Rightarrow$

$pr-cl(E) \cap B = B$(**)

By (*) and (**) we get $B = \phi$ which is a contradiction , since $B \neq \phi$.

Hence F is a pr -connected . Again , since $E \subset pr-cl(E) \subseteq pr-cl(E) \Rightarrow pr-cl(E)$ is a pr -connected .

Theorem (6) :

Let (X, τ, ρ) be a bitopological space , and E is a subset of X . If each two points of E are pr -connected in some

pr -connected subset of E , then E is a pr -connected subset of X .

Proof :

Suppose that E is not pr -connected , then $\exists A, B \neq \phi$ and

$A, B \subset X \ni A \cap pr-cl(B) = \phi$, and $pr-cl(A) \cap B = \phi, E = A \cup B$, since

$A, B \neq \phi \Rightarrow \exists a \in A$ and $b \in B$ such that a and b must be contained in some pr -connected subset F of E since $F \subset A \cup B \Rightarrow$ either $F \subset A$ or $F \subset B \Rightarrow$

either $a, b \in A$ or $a, b \in B$. Let $a, b \in A$, since $b \in B \Rightarrow A \cap B \neq \phi$, which is a contradiction, since A and B are disjoint . Hence E must be a pr -connected .

Theorem (7) :

Let $\{G_\lambda : \lambda \in \Lambda\}$ be a family of pr -connected subsets of a bitopological space (X, τ, ρ) such that $\bigcap \{G_\lambda : \lambda \in \Lambda\} \neq \phi$. Then $\bigcup \{G_\lambda : \lambda \in \Lambda\}$ is a pr -connected set .

Proof :

Suppose that $E = \bigcup \{G_\lambda : \lambda \in \Lambda\}$ is a pr -disconnected $\Rightarrow \exists$ two non-empty disjoint sets G_1 and G_2 both pr -open in the subset E of $X \ni E = G_1 \cup G_2$,

$\forall \lambda, G_1 \cap G_\lambda, G_2 \cap G_\lambda$ are disjoint sets both pr -open in the subset $G_\lambda \ni (G_1 \cap G_\lambda) \cup (G_2 \cap G_\lambda) = (G_1 \cup G_2) \cap G_\lambda = G_\lambda$ (*)

Since G_λ is a pr -connected , one of the sets $G_1 \cap G_\lambda$ and $G_2 \cap G_\lambda$ must be empty , say $G_1 \cap G_\lambda = \phi$, then by (*) $G_2 \cap G_\lambda = G_\lambda \Rightarrow G_\lambda \subset G_2$.

$\forall \lambda \in \Lambda \Rightarrow \cup \{G_\lambda : \lambda \in \Lambda\} \subset G_2 \Rightarrow G_1 \cup G_2 \subset G_2 \Rightarrow G_1 = \phi$

, which is contradiction .

Hence E must be pr -connected since G_1 is non-empty , so E is pr -connected .

Theorem (8) :

Let $\{G_\lambda : \lambda \in \Lambda\}$ be a family of pr -connected subsets of a bitopological space (X, τ, ρ) such that one of the member of this family intersects every other member , then

$\bigcup_{\lambda \in \Lambda} G_\lambda$ is a pr -connected set .

Proof :

Let G_{λ_0} be a fixed member of the given family $\exists G_{\lambda_0} \cap G_\lambda \neq \phi, \forall \lambda \in \Lambda$ then $D_\lambda = G_{\lambda_0} \cup G_\lambda$ is a pr -connected set $\forall \lambda \in \Lambda$ by previous theorem .

Now

$\cup \{D_\lambda : \lambda \in \Lambda\} = \cup \{G_{\lambda_0} \cup G_\lambda : \lambda \in \Lambda\} = G_{\lambda_0} \cup [\cup \{G_\lambda : \lambda \in \Lambda\}] = \cup \{G_\lambda : \lambda \in \Lambda\}$

and

$\cap \{D_\lambda : \lambda \in \Lambda\} = \cap \{G_{\lambda_0} \cup G_\lambda : \lambda \in \Lambda\} = \cap \{D_\lambda : \lambda \in \Lambda\} = G_{\lambda_0} \cup [\cap \{G_\lambda : \lambda \in \Lambda\}] \neq \phi$

(always) ,since by our

assumption $G_{\lambda_0} \neq \phi$ is intersects every $G_\lambda \neq \phi$ as $G_{\lambda_0} \cap G_\lambda \neq \phi, \forall \lambda \in \Lambda$.

Hence by previous theorem $\cup \{D_\lambda : \lambda \in \Lambda\} = \cup \{G_\lambda : \lambda \in \Lambda\}$ is a pr -connected set .

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المجموعات المتصلة تقريبا في الفضاءات ثنائية التبولوجيا

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المستخلص

في هذا البحث نقدم وندرس تعميم لمفهوم المجموعة المتصلة تقريبا في الفضاءات ثنائية التبولوجيا من خلال تقديمنا عدة نتائج متنوعة التي تتعلق بهذا التعميم .