MEROMORPHIC FUNCTIONS THAT SHARE ONE VALUE WITH ITS FIRST DERIVATIVE

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Abstract

In this paper we study the uniqueness of meromorphic functions that share one value only with their derivatives. The results here are improved for the results in [1] and also we gave answer for open question in our paper.

الدوال الميرومورفية التي لها حصة قيمة واحدة مع مشتقتها الأولى

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الخلاصة

في هذا البحث نحن ندرس الوحدانية من الدوال الميرومورفكية التي لها حصة قيمة واحدة فقط مع مشتقاتها، النتائج هنا هي تحسين للنتائج في [1] وكذلك أعطينا جواب للمسألة المفتوحة في بحثنا.

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1. Introduction

Let f be a function meromorphic (i.e. analytic except for poles) and not constant in the complex plane. For any complex a, including ∞, we denote by $n(t,\frac{1}{f-a})$ the number of roots of the equation $f(z)=a$ in $|z|\le t$ $(t\ge 0)$, roots of order p being counted p times, by $n(0,\frac{1}{f-a})$ the order of root of the equation $f(z)=a$ at $z=0$ (if $f(0)\ne a$, then $n(0,\frac{1}{f-a})=0$), by $n(t,\frac{1}{f-\infty}) = n(t,f)$ the number of poles of f in $|z|\le t$, poles of order p being counted p times and by $\bar{n}(t,\frac{1}{f-a})$ the number of distinct roots of $f(z)=a$ in $|z|\le t$. Correspondingly we define

$$N(r,\frac{1}{f-a})=\int_0^r \frac{n(t,\frac{1}{f-a})-n(0,\frac{1}{f-a})}{t}dt+ n(0,\frac{1}{f-a})\log r,$$

$$N(r,f)=\int_0^r \frac{n(t,f)-n(0,f)}{t}dt+ n(0,f)\log r,$$

$$\overline{N}(r,\frac{1}{f-a}) = \int_0^r \frac{\overline{n}(t,\frac{1}{f-a}) - \overline{n}(0,\frac{1}{f-a})}{t} dt + \overline{n}(0,\frac{1}{f-a}) \log r,$$

Let k be a positive integer, we denote by $n_{k)}(t,\frac{1}{f-a})$ ($resp.\ n_{(k+1}(t,\frac{1}{f-a})$) the number of roots of the equation $f(z)=a$ with order $\le k$ ($resp. > k$) counting multiplicities in $|z| \le t$. Similarly as in above, we can define $N_{k)}(r,\frac{1}{f-a})$, $N_{(k+1}(r,\frac{1}{f-a})$, $\overline{N}(r,f)$, $\overline{N}_{k)}(r,\frac{1}{f-a})$, $\overline{N}_{(k+1}(r,\frac{1}{f-a})$, $N_{k)}(r,f)$, $N_{(k+1}(r,f)$, $\overline{N}_{k)}(r,f)$ and $\overline{N}_{(k+1}(r,f)$ (see [2], [3]).

We assume that the reader is familiar with the usual notations and fundamental results of Nevanlinna's theory of meromorphic functions (see [2], [3]). For example,

$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta$, where $\log^+ x = \max\{\log x, 0\}$, $x \ge 0$, $T(r,f) = m(r,f) + N(r,f)$, and $S(r,f)$ will denote any quantity that satisfies $S(r,f) = o(1)T(r,f)$ as $r \to \infty$ possibly outside a set E of r of finite linear measure. We say that two non-constant meromorphic functions f and g share a finite value a IM (ignoring multiplicity), if $f-a$ and $g-a$ have the same zeros. They share a finite value a CM (counting multiplicity), if $f-a$ and $g-a$ have the same zeros with the same multiplicities. And we set

$$N_2(r,\frac{1}{f}) = \overline{N}(r,\frac{1}{f}) + \overline{N}_{(2}(r,\frac{1}{f}).$$

2. The main results

In [4] R. Brück proved the following theorem:

Theorem A.
Let f be a non-constant entire function satisfying $N(r,\frac{1}{f'}) = S(r,f)$. If f and f' share the value 1 CM, then $f-1 = c(f'-1)$, for some nonzero constant c.

In [5] and [6], A. H. H. Al-Khaladi improved Theorem A and proved the following theorems:

Theorem B[5].
Let f be a non-constant meromorphic function satisfying $\overline{N}(r,\frac{1}{f'}) + \overline{N}(r,f) = S(r,f)$. If f and $f^{(k)}$ ($k \ge 1$) share the value 1 CM, then $f-1 =$ $c(f^{(k)}-1)$, for some nonzero constant c.

Theorem C[6].
Let f be a non-constant meromorphic function satisfying $N(r,\frac{1}{f'})$ $= S(r,f)$. If f and f' share the value 1 CM, then $f-1 = c(f'-1)$, for some nonzero constant c.

Theorem C suggests the following question as an open problem:

Question 1. What can be said when a non-constant meromorphic function f shares one nonzero finite value CM with f'?

In this paper, we will answer Question 1. Indeed, we shall prove the following theorems:

Theorem 1.
Let f be a non-constant meromorphic function. If f and f' share the value $a(\ne 0,\infty)$ CM, then one of the following four cases must occur:

(i) $f = f'$.

(ii) $f(z) = \frac{a(z-c)}{1+Ae^{-z}}$, where $A(\ne 0)$ and c are constants.

(iii) $T(r,f) \le 2\overline{N}_{(2}(r,f) + 2N_2(r,\frac{1}{f}) + S(r,f)$.

(iv) $T(r,f) \le 4N_2(r,\frac{1}{f}) + S(r,f)$.

Theorem 2.
Let f be a non-constant meromorphic function. If f and f' share the value $a(0 \ne \infty)$ CM, then either $f = f'$ or

$$T(r,f) \le \overline{N}(r,f) + N_2(r,\frac{1}{f}) + S(r,f).$$

As an immediate consequence of Theorem1, we have

Corollary 1.

Let f be a non-constant meromorphic function. If f and f' share the value a $(0 \neq \infty)$ CM, and if $\overline{N}(r,\frac{1}{f}) = S(r,f)$, then either $f = f'$ or $f(z) = \frac{a(z-c)}{1+Ae^{-z}}$, where $A(\neq 0)$ and c are constants.

This is exactly Theorem 1 in [1].

Theorem 3.

Let f be a non-constant meromorphic function. If f and f' share the value a $(0 \neq \infty)$ IM, then exactly one of the following three cases must occur:

(i) $f = f'$.

(ii) $f(z) = \frac{2a}{1-Ae^{-2z}}$, where A is nonzero constant.

(iii) $T(r,f) \le 4N_2(r,\frac{1}{f}) + 5\overline{N}(r,\frac{1}{f'}) + S(r,f)$.

From Theorem 3, we immediately deduce the following Corollary:

Corollary 2

Let f be a non-constant meromorphic function. If f and f' share the value $a(\neq 0,\infty)$ IM, and if $\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f'}) = S(r,f)$, then either $f = f'$ or $f(z) = \frac{2a}{1-Ae^{-2z}}$, where A is a nonzero constant.

This is exactly Theorem 2 in [1].

3. Proof of Theorem 1

Suppose $a = 1$ (the general case following by considering $\frac{1}{a}f$ instead of f) and $f \neq f'$. We set

$$F = \frac{1}{f}\left(\frac{f''}{f'-1} - \frac{f'}{f-1}\right). \tag{1}$$

From the fundamental estimate of logarithmic derivative it follows that

$$m(r,F) = S(r,f). \tag{2}$$

Suppose z_1 is a simple pole of f. Then the Laurent expansion of f about z_1 is

$$f(z) = a_{-1}(z-z_1)^{-1} + a_0 + a_1(z-z_1) + \cdots,$$

$$(a_{-1} \neq 0) \tag{3}$$

Consequently, from (1),

$$F(z) = \frac{-1}{a_{-1}} + \frac{1}{a_{-1}^2}(z-z_1) + \cdots. \tag{4}$$

Hence

$$F'(z) = \frac{1}{a_{-1}^2} + \cdots. \tag{5}$$

It follows from (4) and (5) that

$$F^2(z_1) - F'(z_1) = 0. \tag{6}$$

Again from (1), if z_p is a pole of f of multiplicity $p \ge 2$, then z_p is possible a zero of F of multiplicity $p-1$, i.e.,

$$F(z) = O((z-z_p)^{p-1}). \tag{7}$$

We consider two cases:

Case 1. $F^2 - F' = 0$. Solving this equation , we have

$$F(z) = \frac{1}{c-z}, \tag{8}$$

where c is a constant. Substituting this into (1) gives

$$\frac{1}{c-z} = \frac{1}{f}\left(\frac{f''}{f'-1} - \frac{f'}{f-1}\right). \tag{9}$$

From this, it is easy to see that

$$N_{(2}(r,f) = 0. \tag{10}$$

If f is a rational function, then $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials have no common zeros. Since f and f' share 1 CM, it follows that the function

$$\frac{f'-1}{f-1} = \frac{P'Q - PQ' - Q^2}{Q(P-Q)}, \tag{11}$$

has no zeros, further, the poles of this function can only occur at the poles of f, i.e., at the zeros of Q. From (10), we know that all zeros of Q are simple, so $P'Q - PQ' - Q^2$ and Q have no common zeros. Thus we conclude from (11) that

$$P'Q - PQ' - Q^2 = c_1(P-Q), \tag{12}$$

where c_1 is a nonzero constant. From (12), (11) and (9) we have $P = (z-c)Q'$. Combining with (12), we arrive at a contradiction. Therefore f is a transcendental meromorphic function, and hence $m(r,c-z)=S(r,f)$. From this and (9), we deduce that $m(r,f)=S(r,f)$. Combining this with (10) yields

$$T(r,f)=N_{1)}(r,f)+S(r,f). \quad (13)$$

Set $$\varphi=\frac{f'-f}{f(f-1)}-F. \quad (14)$$

Then it is clear that

$$m(r,\varphi)\le m(r,\frac{1}{f-1})+m(r,F)+S(r,f),$$

and from (2), we have

$$m(r,\varphi)\le m(r,\frac{1}{f-1})+S(r,f). \quad (15)$$

Since f and f' share the value 1 CM, we see from (14) and (8) that

$$N(r,\varphi)\le \overline{N}(r,\frac{1}{f})+S(r,f). \quad (16)$$

From (15) and (16),

$$T(r,\varphi)\le m(r,\frac{1}{f-1})+\overline{N}(r,\frac{1}{f})+S(r,f). \quad (17)$$

Let z_1 be a simple pole of f. By a simple calculation on the local expansion we see that $\varphi(z_1)=0$. If $\varphi=0$, then from (14) and (8) we conclude that $\frac{d}{dz}[\frac{(z-c)e^z}{f(z)}]=e^z$. By integration and f is a transcendental meromorphic function, we obtain the conclusion (ii). If $\varphi\ne 0$, then

$$N_{1)}(r,f)\le N(r,\frac{1}{\varphi})\le T(r,\varphi)+O(1). \quad (18)$$

Therefore (18), (17) and (13) give that

$$T(r,f)\le m(r,\frac{1}{f-1})+\overline{N}(r,\frac{1}{f})+S(r,f)$$

Hence

$$N(r,\frac{1}{f-1})\le \overline{N}(r,\frac{1}{f})+S(r,f). \quad (19)$$

Set $$H=\frac{f''(f-1)}{f'(f'-1)}. \quad (20)$$

Obviously, $m(r,H)\le m(r,f)+S(r,f)$. Together with (13) we have

$$m(r,H)=S(r,f). \quad (21)$$

It follows from (20) that if z_1 is a simple pole of f, then

$$H(z_1)=2. \quad (22)$$

Since f and f' share 1 CM, we deduce from (20), (22) and (10) that $N(r,H)\le \overline{N}(r,\frac{1}{f'})$. Combining this with (21) we obtain

$$T(r,H)\le \overline{N}(r,\frac{1}{f'})+S(r,f). \quad (23)$$

If $H=2$, we deduce from (20) that $f'-1=c_2(f-1)^2$, with $c_2(\ne 0)$ constant. So f and f' can not share the value 1 CM, which is a contradiction. Thus we conclude $H\ne 2$, and so

$$N_{1)}(r,f)\le N(r,\frac{1}{H-2})\le T(r,H)+O(1)$$
$$\le \overline{N}(r,\frac{1}{f'})+S(r,f), \quad (24)$$

by (23). From the second fundamental theorem, (19), (24) and (10) we have

$$T(r,f)\le N(r,\frac{1}{f})+N(r,\frac{1}{f-1})+\overline{N}(r,f)$$
$$-N(r,\frac{1}{f'})+S(r,f)$$
$$\le N(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f'})-$$
$$N(r,\frac{1}{f'})+S(r,f). \quad (25)$$

Clearly, $N(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f'})-N(r,\frac{1}{f'})$

$$\le N_2(r,\frac{1}{f}). \quad (26)$$

Thus, we find from (25) and (26) that $T(r,f)\le 2N_2(r,\frac{1}{f})+S(r,f)$, and this gives (iii).

Case 2. If $F^2-F'\ne 0$, we deduce from (6), (7) and (2) that $N_{1)}(r,f)+N_{(3}(r,f)$

$$-2\overline{N}_{(3}(r,f)\le N(r,\frac{1}{F^2-F'})\le$$
$$-m(r,\frac{1}{F^2-F'})+T(r,F^2-F')+O(1)$$
$$\le -m(r,\frac{1}{F^2-F'})+N(r,F^2-F')+$$

$S(r,f)$, that is $N_{1)}(r,f)+N_{(3}(r,f)+m(r,\frac{1}{F^2-F'})\le 2\overline{N}_{(3}(r,f)+N(r,F^2-F')+S(r,f)$. (27)

Here we estimate $N(r,F^2-F')$ and $m(r,\frac{1}{F^2-F'})$. Since f and f' share 1 CM, we find from (1), (6) and (7) that the poles of F^2-F' can only occur at the zeros of f. However, the zeros of f with multiplicity $q=1$ ($resp.\ q\ge 2$) are all poles of F^2-F' with multiplicity 2 ($resp$. 4), at most, thus

$$N(r,F^2-F')\le 2N_2(r,\frac{1}{f}). \quad (28)$$

Let h be the function defined by $h=\frac{f'-1}{f-1}$. Then from (1), we have $F=\frac{1}{f}\cdot\frac{h'}{h}$. Hence $f^2=\frac{1}{F^2-F'}[(\frac{h'}{h})^2+\frac{h'}{h}f'-(\frac{h'}{h})'f]$. It follows that

$2m(r,f)\le m(r,\frac{1}{F^2-F'})+m(r,f)+S(r,f)$, that is

$$m(r,f)\le m(r,\frac{1}{F^2-F'})+S(r,f). \quad (29)$$

Combining (27), (28) and (29) we deduce

$$T(r,f)\le 2\overline{N}_{(2}(r,f)+2N_2(r,\frac{1}{f})+S(r,f). \quad (30)$$

This is conclusion (iii).

From (1), (6), (7) and the assumption that f and f' share 1 CM, we see that the poles of F coincide with the zeros of f, in fact the zeros of f with multiplicity $q\ge 1$ are all poles of F with multiplicity at most 2. Thus, we get from (2)

$$T(r,F)\le N_2(r,\frac{1}{f})+S(r,f). \quad (31)$$

If $F\ne 0$, we deduce from (7) and (31) that $\overline{N}_{(2}(r,f)\le N(r,\frac{1}{F})\le T(r,F)+O(1)\le N_2(r,\frac{1}{f})+S(r,f)$. From this and (30), we arrive at the conclusion (iv). If $F=0$, by integrating (1) once gives

$$f'-1=c_3(f-1), \quad (32)$$

with $c_3(\ne 0)$ constant. From this we arrive at (i) or (iv). Thus we complete the proof of Theorem 1.

4. Proof of Theorem 2

If $F\ne 0$, we may obtain from (7) $N_{(2}(r,f)-\overline{N}_{(2}(r,f)\le N(r,\frac{1}{F})\le$

$$\le -m(r,\frac{1}{F})+T(r,F)+O(1). \quad (33)$$

From (1), it follows that

$$m(r,f)\le m(r,\frac{1}{F})+S(r,f). \quad (34)$$

Combining (34), (33) and (31) we find that $N_{(2}(r,f)+m(r,f)\le \overline{N}_{(2}(r,f)+N_2(r,\frac{1}{f})+S(r,f)$. So

$T(r,f)\le \overline{N}(r,f)+N_2(r,\frac{1}{f})+S(r,f)$. If $F=0$, then $\frac{f''}{f'-1}=\frac{f'}{f-1}$. By integration, we obtain (32). Then it is easy to see that either $f=f'$ or $T(r,f)=\overline{N}(r,\frac{1}{f})+S(r,f)$. The proof is complete.

5. Proof of Theorem 3

From (20), we know that if z_∞ is a pole of f of multiplicity $\ell\ge 1$, then

$$H(z_\infty)=\frac{\ell+1}{\ell}. \quad (35)$$

Let z_q be a zero of $f'-1$ of multiplicity $q\ge 1$. Since f and f' share 1 IM, we must have z_q is a simple zero of $f-1$. By a simple calculation on the local expansion we see that

$$H(z_q)=q. \quad (36)$$

From (20), (35) and (36) it can be seen that the poles of H can only occur at the zeros of f'. Thus

$$N(r,H)\le \overline{N}(r,\frac{1}{f'}). \quad (37)$$

Further, if $H\ne 2$, it follows that from (35), (36), (37) and (20) that

$$N_{1)}(r,f)+\overline{N}_{2)}(r,\frac{1}{f'-1})-N_{1)}(r,\frac{1}{f'-1})$$
$$\leq N(r,\frac{1}{H-2})\leq T(r,H)+O(1)$$
$$\leq N(r,H)+m(r,H)+O(1)$$
$$\leq \overline{N}(r,\frac{1}{f'})+m(r,f)+S(r,f). \qquad (38)$$

If z_1 is a simple zero of $f'-1$, then from (1) we find that F will be holomorphic at z_1. If $F\neq 0$, we deduce from this, the hypothesis of Theorem 3, (1), (2), (34) and (7) that

$$N_{(2}(r,f)-\overline{N}_{(2}(r,f)\leq N(r,\frac{1}{F})\leq$$
$$T(r,F)-m(r,\frac{1}{F})+O(1)\leq N(r,F)+$$
$$m(r,F)-m(r,\frac{1}{F})+O(1)\leq N_2(r,\frac{1}{f})+$$
$$\overline{N}_{(2}(r,\frac{1}{f'-1})-m(r,f)+S(r,f). \qquad (39)$$

Combining (39) with (38) yields

$$N(r,f)-\overline{N}_{(2}(r,f)\leq N_2(r,\frac{1}{f})+$$
$$\overline{N}(r,\frac{1}{f'})+\overline{N}_{(3}(r,\frac{1}{f'-1})+S(r,f). \qquad (40)$$

This implies that $\overline{N}(r,f)\leq N_2(r,\frac{1}{f})+$

$$\overline{N}(r,\frac{1}{f'})+\overline{N}_{(3}(r,\frac{1}{f'-1})+S(r,f).$$

From this and the second fundamental theorem for f', we find that

$$T(r,f')\leq N(r,\frac{1}{f'})+N(r,\frac{1}{f'-1})+$$
$$\overline{N}(r,f)-N(r,\frac{1}{f''})+S(r,f)\leq N(r,\frac{1}{f'})$$
$$+N(r,\frac{1}{f'-1})+N_2(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f'})$$
$$+\overline{N}_{(3}(r,\frac{1}{f'-1})-N(r,\frac{1}{f''})+S(r,f).$$

Hence it follows that

$$N(r,\frac{1}{f''})\leq N(r,\frac{1}{f'})+N_2(r,\frac{1}{f})+$$
$$\overline{N}_{(3}(r,\frac{1}{f'-1})+\overline{N}(r,\frac{1}{f'})+S(r,f). \qquad (41)$$

Obviously $\overline{N}_{(3}(r,\frac{1}{f'-1})+\overline{N}_{(2}(r,\frac{1}{f'-1})$

$$+N(r,\frac{1}{f'})\leq N(r,\frac{1}{f''})+\overline{N}(r,\frac{1}{f'}). \qquad (42)$$

Then (41) and (42) imply $\overline{N}_{(2}(r,\frac{1}{f'-1})$

$$\leq N_2(r,\frac{1}{f})+2\overline{N}(r,\frac{1}{f'})+S(r,f). \qquad (43)$$

On the other hand, by (39) we get

$$\overline{N}_{(2}(r,f)+m(r,f)\leq N_2(r,\frac{1}{f})+$$
$$\overline{N}_{(2}(r,\frac{1}{f'-1})+S(r,f). \qquad (44)$$

Combining (40), (44) and (43) we obtain

$$T(r,f)\leq 4N_2(r,\frac{1}{f})+5\overline{N}(r,\frac{1}{f'})+S(r,f).$$

This is the conclusion (iii).

If $F=0$, then similar as the proof of Theorem 2, we will arrive at (i) or (iii).

If $H=2$, then we find from (20) that

$$f'-1=c(f-1)^2, \qquad (45)$$

where c is a nonzero constant. We rewrite (45) in the form $f'-1=$

$c(f-1+A)(f-1-A)$, where $A^2=-\frac{1}{c}$.

Hence $\overline{N}(r,\frac{1}{f'})=\overline{N}(r,\frac{1}{f-1+A})+$

$\overline{N}(r,\frac{1}{f-1-A})$. It follows from the second fundamental theorem for f that if $A\neq\pm 1$,

$$T(r,f)\leq\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-1+A})+$$
$$\overline{N}(r,\frac{1}{f-1-A})+S(r,f)$$

$\leq\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f'})+S(r,f)$, which is (iii). If $A=\pm 1$, we have $A^2=1$ and so $c=-1$. Thus (45) reads $\frac{f'}{f-2}-\frac{f'}{f}=-2$.

By integration once we conclude (ii). This proves Theorem 3.

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References

1. Al-Khaladi, A. H. H. **2009**. Meromorphic functions that share one finite value CM or IM with their first derivative, *Journal of Al-Anbar university for pure science.* **3**(3): 69-73.
2. Hayman, W. K. **1964**, *Meromorphic functions*. Clarendon Press, Oxford. pp.1-42.
3. Yi, H. X. and Yang, C. C. **1995**, *Uniqueness theory of meromorphic functions*. Science Press, Beijing. (In Chinese) pp.1-25.
4. Brück, R. **1996**. On entire functions which share one value CM with their first derivative, *Result. Math*. **30**: 21-24.
5. Al-Khaladi, A. H. H. **2003**. On two meromorphic functions that share one value CM, *Kodai Math. J*. **26** (3): 259-266.
6. Al-Khaladi, A. H. H. **2005** On meromorphic functions that share one value with their derivative, *Analysis* **25**:131-140.