About Fractional Operators: Behavior and Extension
Alauldin Noori Ahmed, Ahmed Ayyoub Yousif

Al-Nahrain University - College of Science - Department of Mathematic & Computer Applications – Iraq

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Abstract
In this paper, we are study some behavior fractional operators solutions, such as existence, uniqueness, stabilities and extended formulas for a dynamic multi-fractional order differential equations. Theorems are stated and proved based on the definitions of Riemann-Liouville fractional integral and Grunwald-Letnikov fractional derivative. Also, a simple numerical approach has been implemented to solve such system.

Key Words: Fractional Calculus, Existence and Uniqueness solution, Stability.
1. Introduction

As to the history of fractional calculus [10], already in 1695 L’Hospital raised the question as to the meaning of \( \frac{d^n y}{dx^n} \) if \( n = \frac{1}{2} \), that is “what if \( n \) is fractional”. “This is an apparent paradox from which, one day, useful consequences will be drawn”, Leibniz replied, together with “\( \frac{d^n x}{dx^n} \) will be equal to \( \sqrt{dx} : x \)”. Lacroix, was the first to mention in some two pages a derivative of arbitrary order in a 700 page text book of 1819. Although fractional derivatives have a long mathematical history, for many years they were not used in many different sciences, but in recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. Recently, there has been some attempt to solve linear problems with multiple fractional derivatives problems, see [2]. Not much has been done for the nonlinear problems. Problems of stability appeared for the first time in mechanics during the investigation of an equilibrium state of a system. A simple reflection may show that some equilibrium state of a system is stable with respect to small perturbations. The existing methods developed so far for stability check are mainly for integer order systems. However, for the fractional order systems, it is difficult to evaluate the stability by simple examining its characteristic equation either by finding it’s dominate roots or by using other algebraic methods. Direct check of the stability of fractional order system using polynomial criteria is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudo-polynomial function of fractional powers of the complex variables, see [2]. The study of stability of such systems focuses a great interest. We can cite in this domain, the works in [12] and [13] for the stability of linear fractional systems, while the works in [9], and [11] are for the stability of fractional systems with time delay.

This paper, is organized in the following, as well as the introduction in section one. The preliminaries and notations of the problem study is presented in section two, by recalling the definitions of the left and right fractional derivatives of Riemann-Liouville and Gunwald-Letnikov fractional derivatives. The existence and uniqueness solution theorem is stated and proved in section three. Theorems for the stability solution on finite and infinite time interval are stated and proved, in section four. In section five, other stability theory is presented for different structure fractional systems. In section six, the left and right fractional derivatives operator \( D^{a, \beta} \), which have satisfy a semi-group property and the adequate functional spaces
on which, a previous work of Erwin and Roop [2], can be extended, to discuss some properties and cases, such as fractional stationary in the fractional order dynamic systems, to keep its track information of the past and future.

In this paper, we will consider the system of nonlinear multi-fractional order differential equations

\[ y_i^{(n_i)}(t) = f_i(t, Y(t)), \quad i=1, \ldots, m \]  

(1.1)

where \( Y(t) = (y_1(t), \ldots, y_m(t))^T \) is the solution of the system (1), and \(0 < \alpha_i \leq 1\), in which two main definitions of Riemann-Liouville fractional integration, and Grunwald-Letnikov fractional derivative are used.

2. Definitions

([8], [9], [10], and [11]).

**Definition 1:** (Left and Right Riemann-Liouville Fractional integral) Let \( y \) be a function defined on \((a, b)\), and \( \alpha > 0 \). Then the left and right Riemann-Liouville fractional integral of order \( \alpha \) are defined to be

\[ _a D_t^{-\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \]

\[ _b D_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} y(s) ds \]  

(2.1)

Left and right (RL) integrals satisfy some important properties like the semi-group property. For more details, we refer to [3].

**Definition 2:** (Left and Right Riemann-Liouville fractional derivative) Let \( \alpha > 0 \), the left and right Riemann-Liouville derivative of order \( \alpha \), denoted by \( _a D_t^\alpha \) and \( _b D_t^\alpha \) respectively, are defined by

\[ _a D_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\alpha-1} y(s) ds, \]

\[ _b D_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_t^b (t-s)^{n-\alpha-1} y(s) ds \]  

(2.2)

where \( n=[\alpha]+1 \), such that \( n-1 \leq \alpha < n \).
The traditional definition of the derivative of a continuous function is expressed in terms of a limiting procedure. Consider the Taylor expansion of the continuous function \( Y(t) \) given by

\[
Y(t + \delta) = Y(t) + \delta D Y(t) + \frac{\delta^2}{2!} D^2 Y(t) + \cdots + \frac{\delta^n}{n!} D^n Y(t) + \cdots
\]

where \( D \) is the derivative operator and writing the upper shift operator, \( E_\delta \), as \( E_\delta Y(t) = Y(t + \delta) \), we obtain \( E_\delta Y(t) = e^{\delta D} Y(t) \). Therefore, we have \( E_\delta = 1 - \Delta_\delta = e^{\delta D} \), so that the right-difference operator becomes

\[
\Delta_- = 1 - E_\delta = 1 - e^{\delta D}.
\]

In this way we have the definition of the derivative operator

\[
\lim_{\delta \to 0} \frac{\Delta_\delta}{\delta} = -D
\]

it is clear from the above, that if \( t \) is the time derivative operator and when applied to a continuous function of time yields

\[
\frac{d}{dt} Y(t) = -\lim_{\delta \to 0} \frac{\Delta_\delta Y(t)}{\delta},
\]

we can do a similar construction for the downshift operator \( E^{-\delta} Y(t) = Y(t - \delta) \), yields

\[
\lim_{\delta \to 0} \frac{\Delta_\delta}{\delta} = D
\]

So (2.4) and (2.5) are equivalent mathematical expressions of the derivative operator. To be on safe side, we write the derivative operator in the symmetric form

\[
\lim_{\delta \to 0} \frac{\Delta_\delta - \Delta_-}{2\delta} = D
\]

Now, we extend the definition of integer-order derivatives to that of fractional order by generalizing the definition of the limit of finite differences. Consider (2.3) raised to the \( \alpha \)th power

\[
\Delta_\delta^\alpha = (1 - e^{-\delta D})^\alpha
\]

so that we can write for the left-side fractional time derivative

\[
\lim_{\delta \to 0} \frac{\Delta_\delta^\alpha}{\delta^\alpha} Y(t) = \lim_{\delta \to 0} \frac{(1 - e^{-\delta D})^\alpha}{\delta^\alpha} Y(t) = \frac{d^\alpha}{dt^\alpha} Y(t),
\]

where we have replaced \( D \) with the time-derivative operator. In a similar way we can write

\[
\lim_{\delta \to 0} \frac{\Delta_-^\alpha}{\delta^\alpha} Y(t) = \lim_{\delta \to 0} \frac{(1 - e^{-\delta D})^\alpha}{\delta^\alpha} Y(t) = \left(-\frac{d^\alpha}{dt^\alpha}\right)^\alpha Y(t),
\]
for the right-side fractional derivative. We can use (2.6) and (2.7), to write the more general formal expression for the left-side and right-side fractional derivative operators as

$$D_{\pm}^{\alpha}Y(t) = \lim_{\delta \to 0} \frac{\Delta_{\pm}^{\alpha}Y(t)}{\delta^{\alpha}}$$  \hspace{1cm} (2.8)

which is mentioned in [10], as the Grunwall-Letnikov fractional derivative

$$\Delta_{\pm}^{\alpha}y(x) = \sum_{j=0}^{\infty} (-1)^{j} \binom{\alpha}{j} y(x - j\delta)$$  \hspace{1cm} (2.9)

which is coincides with the classical $r^{th}$ right handed difference $\Delta_{r}^{\alpha}y(x)$ for $\alpha = r \in \mathbb{N}$, (Note that $\begin{pmatrix} r \\ j \end{pmatrix} = 0$ for $j \geq r+1$). The existence of $D_{\pm}^{\alpha}y$ in the uniform norm is equivalent to the fact that the $r^{th}$ ordinary derivative $y^{(r)}(x)$ exists for all $x$ and is continuous, and this suggests that $D_{\pm}^{\alpha}y$ can be handled by considering the limit in norm. Thus, $D_{\pm}^{\alpha}y$ will be function $g$ in $C_{2\pi}$ or $L_{p,2\pi}^{p}$, $1 \leq p < \infty$, respectively, for which

$$\lim_{\delta \to 0} \|h^{\alpha} \Delta_{\pm}^{\alpha}y(x) - g\| = 0$$

exists. Therefore, we can write the fractional derivative in the Grunwall-Letnikov sense as

$$D_{\pm}^{\alpha}y(t) = \lim_{\delta \to 0} (\delta^{\alpha}) \sum_{j=0}^{\left[ t \right]} (-1)^{j} \binom{\alpha}{j} y(t - j\delta) = \sum_{j=0}^{\left[ t \right]} \binom{\alpha}{j} y(t - j\delta)$$  \hspace{1cm} (2.10)

where $[t]$ is the integer part of $t$, $\delta$ is the step size, and

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha + 1)}{\Gamma(j + 1)\Gamma(\alpha + 1 - j)} = \frac{\alpha!}{j!(\alpha - j)!}$$

The definition of operator in the Grunwald-Letnikov sense (2.10) is equivalent to the definition of operator in the Riemann-Liouville sense (2.1). Nevertheless, the Grunwald-Letnikov operator is more flexible and most straightforward in numerical calculations, in which, the solution of (2.10) is given by the following recurrence equation: (see [22])

$$y_{0} = \beta, \quad y_{n} = f(t_{n}, y_{n}) - \sum_{j=1}^{n} c_{j}^{\alpha} y_{n-j}, \quad (n=1,2,3, \ldots),$$  \hspace{1cm} (3.1)

where $t_{n} = n\delta$, $y_{n} = y(t_{n})$ are positive functions for all $n$, and $c_{j}^{\alpha}$ are the Grunwald-Letnikov coefficients defined as:

$$c_{0}^{\alpha} = \delta^{\alpha}, \quad c_{j}^{\alpha} = \left(1 - \frac{\alpha}{j}\right) c_{j-1}^{\alpha}, \quad (j=1,2,3, \ldots)$$

for more details, see [9].

### 3. Existence and Uniqueness Solution:

By Approximating the fractional derivative in (1) by (2.10), leads to the numerical solution algorithm described by (2.11), in the following recursive relations [10]:

$$y_{1,0} = \beta, \quad y_{i,n} = f(t_{i,n}, y_{i,n}) - \sum_{j=1}^{i} c_{j}^{\alpha} y_{i,n-j}, \quad (n=1,2, \ldots),$$  \hspace{1cm} (3.1)
this algorithm is simple for computational performance for all values of $\alpha$, $0 < \alpha \leq 1$.

For details about the fractional difference method and its applications for solving fractional differential equations, we refer the reader to [11].

Set $\hat{D} = I \times C^*(t)$ where $C^*(t)$ is the class of all continuous column vectors $Y(t)$ with the norm

$$\|Y(t)\| = \sum_{i=1}^{m} \|y_i(t)\| = \sum_{i=1}^{m} \max_{I} |y_i(t)|$$

Now, we can state the following theorem:

**Theorem (1):** Let $F(t,Y(t)) \in C^*(\hat{D})$, where $F(t,Y(t)) = (f_1(t,Y(t)), \ldots, f_m(t,Y(t)))^T$, i.e. $f_i(t,Y(t)) \in C(\hat{D})$ for all $i=1,\ldots,m$, and each satisfies the Lipschitz condition

$$|f_i(t,Y(t)) - f_i(t,X(t))| \leq k \sum_{j=0}^{m} |y_j(t) - x_j(t)|$$

for all $i$, (3.2)

$t,Y(t), (t,X(t)) \in \hat{D}$ and $k = \min k_i > 0$.

If $\max C^\alpha_j \leq (1 - k)$, (3.3)

then (1) has one and only one solution $Y(t) \in C(I)$ that satisfies $D^\alpha Y(t) \in C(I)$.

**Proof:**

From (2.11), if we write $TY_n = F(t,Y_n(t)) - \sum_{j=1}^{m} C^\alpha_j Y_{n-j}$, then for $(t,Y(t)),(t,X(t)) \in \hat{D}$, we get

$$\|TY_n - TX_n\| = \left\| F(t,Y_n(t)) - \sum_{j=1}^{l} C^\alpha_j Y_{n-j} - \left( F(t,X_n(t)) - \sum_{j=1}^{l} C^\alpha_j X_{n-j} \right) \right\|$$

$$\leq \left\| (F(t,Y_n(t)) - F(t,X_n(t))) + \left( \sum_{j=1}^{l} C^\alpha_j Y_{n-j} - \sum_{j=1}^{l} C^\alpha_j X_{n-j} \right) \right\|$$

$$\leq k \sum_{j=0}^{m} |y_j(t) - x_j(t)| + \sum_{j=1}^{l} \max |C^\alpha_j| \|Y_{n-j} - X_{n-j}\|$$

$$\leq k \|Y_n - X_n\| + (1-k) \|Y_{n-j} - X_{n-j}\|$$

$$= \|Y_n - X_n\|, \text{ as } n \to \infty, Y_{n-j} \sim Y_n \text{ and } X_{n-j} \sim X_n.$$

Hence, the mapping $T : C^*(\hat{D}) \to C^*(\hat{D})$ is a contraction mapping, and then it has a fixed point $Y(t) = T(Y(t))$.

Providing the condition (3.3) and hence, there exists a unique solution $Y(t) \in C^*(\hat{D})$ for the system (1).$

**4. Stability Solution**

First, we set $I = [0,t_f]$, say, $t_f$ is a finite suitable positive number, and define the norm

$$\|X(t)\| = \sum_{i=1}^{m} |x_i(t)|$$

Now, we can state the following theorem:

**Theorem (2):** Let $F(t,Y(t)) \in C^*(\hat{D})$, where $F(t,Y(t)) = (f_1(t,Y(t)), \ldots, f_m(t,Y(t)))^T$, i.e. $f_i(t,Y(t)) \in C(\hat{D})$ for all $i=1,\ldots,m$, and each satisfies the Lipschitz condition

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$$\leq \left\| (F(t,Y_n(t)) - F(t,X_n(t))) + \left( \sum_{j=1}^{l} C^\alpha_j Y_{n-j} - \sum_{j=1}^{l} C^\alpha_j X_{n-j} \right) \right\|$$

$$\leq k \sum_{j=0}^{m} |y_j(t) - x_j(t)| + \sum_{j=1}^{l} \max |C^\alpha_j| \|Y_{n-j} - X_{n-j}\|$$

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$$\leq k \sum_{j=0}^{m} |y_j(t) - x_j(t)| + \sum_{j=1}^{l} \max |C^\alpha_j| \|Y_{n-j} - X_{n-j}\|$$

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\[ \mathcal{I}_0^\alpha \|Y(t)\| = \sum_{i=1}^{m} \mathcal{I}_0^\alpha \|v_i(t)\| \]  
(4.1)

where

\[ \mathcal{I}_0^\alpha \|v_i(t)\| = \int_{0}^{t} \|v_i(t)\| dt \]  
(4.2)

Riemann-Liouville fractional integration of order \( \alpha_i \) is defined as:

\[ y_i(t) = \mathcal{I}_0^\alpha f_i(t, Y(t)) = \frac{\alpha_i(t-t_0)^{\alpha_i}}{1\Gamma(\alpha_i)} + \frac{1}{1\Gamma(\alpha_i)} \int_{0}^{t} (t-s)^{\alpha_i-1} f(s, Y(s)) ds \]  
(4.3)

In order to study the \( L_1 \)-stability solution of the system (1), we must prove that each \( y_i(t), (i=1, \ldots, m) \) given by (4.3) is bounded, i.e. \( \|y_i(t)\| < \infty \), for all \( i=1, \ldots, m \).

We define the convolution operator of \( f \) and \( g \) as follows:

\[ f \otimes g(t) = \int_{0}^{t} f(t-s) g(s) ds \]  
(4.4)

therefore, the solution \( y_i(t) \) can be rewritten using the product operator \( \otimes \) as follows

\[ y_i(t) = (k_{\alpha_i} \otimes f_i)(t) \]  
(4.5)

Where \( k_{\alpha_i} \) with \( 0 < \alpha_i \leq 1 \) is so called convolution kernel defined by

\[ k_{\alpha_i} = \frac{e^{\alpha_i}}{1\Gamma(\alpha_i)} \]  
(4.6)

**Theorem (2):** Let \( k_{\alpha_i}, f_i \in C([R^+, R^+] \cap L_1([0, t_f])) \), with \( t_f > 0 \) and finite, then the solution of (1) is stable, providing that

\[ \left| \frac{e^{\alpha_i}}{\Gamma(\alpha_i)} \right| < \varepsilon \]  
(4.7)

**Proof:**

With no loss of generality we can take \( y_0 = 0 \) and \( t_0 = 0 \). This reduces (4.3) into

\[ y_i(t) = \frac{1}{1\Gamma(\alpha_i)} \int_{0}^{t} (t-s)^{\alpha_i-1} f(s, Y(s)) ds \]

**First**, we prove that \( \mathcal{I}_0^\alpha \|Y(t)\| < \infty \).

We have

\[ \mathcal{I}_0^\alpha \|Y(t)\| = \sum_{i=1}^{m} \mathcal{I}_0^\alpha \|v_i(t)\| = \sum_{i=1}^{m} \left( \int_{0}^{t} (k_{\alpha_i} \otimes f_i)(t) dt \right) \]

So, we need only to prove that \( \mathcal{I}_0^\alpha \|v_i(t)\| < \infty \)

\[ \mathcal{I}_0^\alpha \|v_i(t)\| = \int_{0}^{t} k_{\alpha_i} \otimes f_i(t, Y(t)) dt \]

\[ = \int_{0}^{t} \int_{0}^{t} k_{\alpha_i} (t-s) f_i(s, Y(s)) ds dt \]  
(by Fubini Theorem for positive functions)
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\[
\int_0^t f_i(s, Y(s)) \left( \int_0^{t-s} k_{a_i}(t-s)d(t-s) \right) ds \quad \text{(by using the change of variables)}
\]

\[
= \int_0^t f_i(s, Y(s)) \left( \int_0^{t-s} k_{a_i}(t)dt \right) ds,
\]

since all \( k_{a_i} \) is positive, we have \( \int_0^{t-s} k_{a_i}(t)dt \leq \int_0^t k_{a_i}(t)dt \)

and \( t_0' \| y_i(t) \| \leq \left( \int_0^t k_{a_i}(t)dt \right) \left( \int_0^\infty f_i(s, Y(s))ds \right) \)

since \( t_f \) is finite, then the right side of the inequality is converge, and we get

\[
\int_0^t y_i(t) dt \leq \left( \frac{(t_f)^{\alpha_i}}{\alpha_i \Gamma(\alpha_i)} \right) \int_0^\infty f_i(s, Y(s))ds < \infty
\]

providing that, the condition (4.7) is satisfied.

**Second**, if we consider two solutions \( Y(t) \) and \( X(t) \) with different initial points \( Y_0 \) and \( X_0 \) with \( \| Y_0 - X_0 \| < \delta(t_f) \). Perform the same above steps for \( t_0' \| Y(t) - X(t) \| \)

\[
\int_0^t \left( \int_0^{t-s} k_{a_i}(t-s)d(t-s) \right) ds < \infty
\]

As we have seen from (4.8), the second term of (4.9) is converge and bounded. Also the first term of (4.9) is converge and bounded, providing that condition (4.7) is satisfied.

Therefore we get

\[
\int_0^t \| Y(t) - X(t) \| < \infty.
\]

The generalization to the infinite case where \( t \in \mathbb{R}^+ \) is not possible because the kernel \( k_{a_i}(t) \) does not belong to \( L_1(\mathbb{R}^+) \), and then \( \int_0^\infty k_{a_i} \otimes f_i(t, Y(t))dt \) does not converge, and

\[
\int_0^\infty k_{a_i} \otimes f_i(t, Y(t))dt \leq \left( \int_0^\infty f_i(s, Y(s))ds \right) \left( \int_0^\infty k_{a_i}(t)dt \right) \text{ does not true.}
\]

Also, even if the system is defined since \( t_0=0 \), the generalization to the infinite case where \( t \in [t_0, \infty) \) is not possible, since the solution of the system (1) given by (4.3) can’t be defined by using a convolution product which is commutative. However, our kernel function given by (4.6) does not depend on the initial time. Therefore, in order to overcome the problems difficulties, we consider our attention to the formal derivation of fractional derivatives using the continuum limit of finite difference equations.
The definition of operator in the Grunwald-Letnikov sense (2.10) is equivalent to the definition of operator in the Riemann-Liouville sense (2.2). Nevertheless, the Grunwald-Letnikov operator is more flexible and most straightforward in numerical calculations, (see [22]).

If we denote \( y_i(t) = \lim_{n \to \infty} y_n \), for all \( i = 1, \ldots, m \), and \( f(t, Y(t)) = \lim_{n \to \infty} f(t, Y(t_n)) \).

Define the norm of \( Y(t) \) as following

\[
\|Y(t)\| = \sum_{i=1}^{m} \|y_i(t)\| \tag{4.10}
\]

**Theorem (3):** The solution of (1) is asymptotically stable, providing that \( f_i \in L^p[0,\infty) \), with \( p \geq 1 \), (for all \( i = 1, \ldots, m \)).

**Proof:**

Consider the solution (3.1).

**First,** we will prove that the system (1) is stable.

\[
\|y_i(t)\| = \left\| f_i(t, Y(t)) - \sum_{j=1}^{n} c_j^\alpha y_i(t_{n-j}) \right\|
\]

Since \( 0 < c_j^\alpha < 1 \), for all \( j = 1, 2, \ldots, n \), and all \( 0 \leq y_i(t_{n-j}) \leq \beta \), then all \( \lim_{n \to \infty} \sum_{j=1}^{n} c_j^\alpha y_i(t_{n-j}) \) is converge and bounded, and since \( f_i \in L^p(0,\infty) \), with \( p \geq 1 \), (for all \( i = 1, \ldots, m \)), the right side is bounded, and we have \( \|y_i(t)\| < \infty \), and then \( \|Y(t)\| < \infty \).

**Second,** if we consider two solutions \( Y(t) \) and \( X(t) \) with two different initial values \( Y_0 \) and \( X_0 \), with \( |y_i(t_0) - x_i(t_0)| < \epsilon \), for all \( i = 1, \ldots, m \), and \( 0 < \epsilon < 1 \), then \( \lim_{n \to \infty} \left( y_i(t_{n-j}) - x_i(t_{n-j}) \right) \to 0 \) and \( \lim_{n \to \infty} \left( f_i(t, Y(t_n)) - f_i(t, X(t_n)) \right) \to 0 \) as \( n \to \infty \).

Performing same as above steps, we get \( \|Y(t) - X(t)\| \to 0 \), as \( t \to \infty \).

Therefore, the solution of the system (1) is asymptotically stable. \( \blacksquare \)

5. Other Stability Approach:

([12], [17], [20], [23] and [24])

As we mentioned before in this paper, exponential stability cannot be used to characterize asymptotic stability of fractional order systems. Many general fractional order systems can be described (or transform) in the following form

\[
G(s) = \frac{b_m s^\beta_m + \cdots + b_k s^\beta_k + b_0 s^\beta_0}{a_m s^{\alpha_m} + \cdots + a_k s^{\alpha_k} + a_0 s^{\alpha_0}} = \frac{Q(s^\beta)}{P(s^{\alpha})}, \tag{5.1}
\]

where \( a_k (k=1, \ldots, m) \), \( b_k (k=1, \ldots, n) \) are constant; and \( \alpha_k (k=1, \ldots, n) \), \( \beta_k (k=1, \ldots, m) \) are arbitrary real numbers and without loss of generality they can be arranged as \( \alpha_0 > \alpha_{r+1} > \cdots > \alpha_0 \) and \( \beta_m > \beta_{m-1} > \cdots > \beta_0 \). Also, the form (5.1) can be put into the following form
with \( a_k = \alpha_k \), \( \beta_k = \alpha_k \), \((0 < \alpha < 1)\). \( \forall k \in \mathbb{Z}, N > M \) and \( \lambda_i \) \((i=1,\ldots,N)\) are the roots of pseudo-polynomial \( P(s^\alpha) \) or the system poles which are assumed to be simple without loss of generality. The analytical solution of the system (5.2) can be expressed as

\[
y(t) = L^{-1}\left\{ K_0 \left[ \sum_{i=1}^{N} \frac{A_i}{s^{\alpha_i} + \lambda_i} \right] \right\} = K_0 \sum_{i=1}^{N} A_i t^\alpha E_{\alpha,\alpha}(-\lambda_i t^\alpha). \tag{5.3}
\]

where \( E_{\alpha,\alpha}(.) \) is the Mittag-Leffler function of two parameters \([10]\).

Several authors using several methods that a geometrical method of complex analysis based on the argument principle of the roots of the characteristic equation can be used for the stability check. The stability condition can be stated as follows

**Theorem (4):** The system (5.2) is stable if and only if \(|\arg(\lambda_i)| > \alpha \frac{\pi}{2}\), for all \( i \) with \( \lambda_i \) is the \( i \)-th root of \( P(s^\alpha) \).

A new definition was introduced in \([20]\), can be presented as follows:

**Definition 4:** The trajectory \( y(t)=0 \) of the system (1) \( t^\gamma \) is asymptotically stable if there is a positive real \( r \) such that: \( \forall \| y(t) \| \) with \( t \leq t_0 \), \( \exists N(y(t)) \), such that \( \forall t \geq t_0 \), \( \forall \| y(t) \| \leq N t^\gamma \).

The fact that the components of \( y(t) \) slowly decay towards 0 following \( t^\gamma \) leads to fractional systems sometimes being called long memory systems. Power law stability \( t^\gamma \) is a special case of the Mittage-Leffler stability \([18]\). According to the stability theorem defined in \([24]\), the equilibrium points are asymptotically stable for \( r_1 = r_2 = \cdots = r_n = r \) if all the eigenvalues \( \lambda_i \) \((i=1,\ldots,n)\) of the Jacobian matrix \( J = \frac{\partial f}{\partial y} \) evaluated at the equilibrium, satisfy the condition \(|\arg(\lambda_i)| > r \frac{\pi}{2}, i=1,\ldots,n \) \([23]\).

If \( r_1 \neq r_2 \neq \cdots \neq r_n \) and suppose that \( m \) is the LCM of the denominators \( u_i/s \) of \( r_i/s \), where \( r_i = \frac{u_i}{s} \), \( v_i, u_i Z^+ \) for \( i=1,\ldots,n \) and setting \( \gamma = \frac{1}{m} \). System (1) is asymptotically stable if \(|\arg(\lambda)| > r \frac{\pi}{2}, \) for all roots \( \lambda \) of the equation

\[
\det(diag(\lambda^{u_1}, \lambda^{u_2}, \ldots, \lambda^{u_n})) - J = 0, \tag{5.4}
\]

A necessary stability condition for the system (1) to remain chaotic is keeping at least one eigenvalue \( \lambda \) in the unstable region \([23]\). The above theory can demonstrated by the following nonlinear fractional order system:

\[
D_t^{\alpha_1} y_1(t) = 35[y_2(t) - y_1(t)] \\
D_t^{\alpha_2} y_2(t) = -7y_1(t) - y_1(t)y_3(t) + 28y_2(t) \\
D_t^{\alpha_3} y_3(t) = y_1(t)y_2(t) - 3y_3(t)
\]

The system has three equilibrium at \((0,0,0)\), \((7.94,7.94,21)\), and \((-7.94,-7.94,21)\). The Jacobian matrix of the system evaluated at \((y^*_1, y^*_2, y^*_3)\) is:

\[
\begin{bmatrix}
-35 & 35 & 0 \\
-7 - y_3^* & 28 & -y_1^* \\
y_2^* & y_1^* & -3
\end{bmatrix},
\]

The two last equilibrium points are saddle points and (5.4) becomes as follows:

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\[ \lambda^{27} + 35 \lambda^{19} + 3 \lambda^{18} - 28 \lambda^{17} + 105 \lambda^{10} - 21 \lambda^{8} + 4410 = 0 \]

And this characteristic equation has unstable roots \( \lambda_{1,2} = 1.2928 \pm 0.2032j \), \( |\text{arg}(\lambda_{1,2})| = 0.1560 \), and therefore, the system satisfy the stability condition.

6. Extended Fractional Operator [4]:

Looking for the past \( P_t \) and the future \( F_t \) of a given dynamical process \( x(s) \), \( s \in \mathbb{R} \) at time \( t \), i.e. on the information \( P_t = \{ x(s), a \leq s \leq t \} \) and \( F_t = \{ x(s), t \leq s \leq b \} \) where \( a \) and \( b \) can be chosen and depends on the amount of information we are keeping from the past and the future. This induces the fact that we look for two quantities, not yet define that we denote by \( d^-_x(t) \) and \( d^+_x(t) \) from the point of view of derivatives. The past and future information can be weighted, achieved by using a weight \( \frac{1}{t-f} \) and regularizing the corresponding function.

We then are lead to two quantities \( d^-_x(t) \) and \( d^+_x(t) \), which represent weighted information on the past and future behavior of the dynamical process.

The previous idea is well formalized by the left and right (Riemann-Liouville) derivatives [18]. We have used in this paper the left and right Riemann-Liouville derivatives with different indices for the left and right differentiation, i.e. we consider \( D^\alpha_a \) and \( D^\beta_b \). The extended operator depends naturally on these two operators and is denoted \( D^{\alpha,\beta} \). However, this operator does not reduce to the ordinary derivative only when \( \alpha = \beta = 1 \). In [1] Agrawal has studied fractional variation problems using the Riemann-Liouville derivatives. He notes that even if the initial functional problems only deals with the left Riemann-Liouville derivative, the right Riemann-Liouville derivative appears naturally during the computations. Several useful functional spaces were introduced. Let \( I \subseteq \mathbb{R} \) be an open interval (which may be unbounded). We denote by \( C_c^\infty(I) \) the set of all functions \( x \in C^\infty(I) \) that vanish outside a compact subset \( K \) of \( I \). In the classical integrals, and derivatives of order one, are obtained by setting \( \alpha = 1 \). In this section, the fractional derivative operator of order \((\alpha, \beta)\) introduced in [4], is presented in the following form:

\[
D\nu^{\alpha,\beta} = \frac{1}{\Gamma(1-\nu)} \left[ \left( \frac{D^\alpha_a}{D^\beta_b} \right) \right]_{t-a}^{t-b} + i\mu \left( \frac{D^\alpha_a}{D^\beta_b} \right) \] (6.1)

where \( a, b \in \mathbb{R}, a < b, \mu \in \mathbb{C} \) and \( \alpha, \beta > 0 \), and its generalization present in the following generalized definition, based on differentiating fractional integrals.

Definition 5: The (right-left-sided) fractional derivative of order \( 0 < \alpha < 1 \) and type \( 0 \leq \beta \leq 1 \) with respect to \( t \) is defined by

\[
D^{\alpha,\beta}_{a+} y(t) = \left[ \pm \int_{a+}^{t} \left( \int_{a+}^{t} \right)^{1-\beta}(t-x)^{\alpha-1} y \right]^{\beta} \] (6.2)

for functions for which the expression on the right hand side exists.

The above definition is equivalent to definition of right and left Riemann-Liouville fractional derivative, when \( \beta = 0 \) and 1, which seems that the fractional derivatives of general type \( 0 < \beta < 1 \) have not been considered. To explain the implementation of such operator, it can
be possible to discuss the infinitesimal form of fractional stationary, by considering the following equation

\[ D^{\alpha,\beta}_{\alpha,\beta} y(t) = 0 \]  

(6.3)

with initial condition

\[ I_{\alpha,\beta}^{\alpha,\beta(\alpha - 1)} y(0+) = y_0 \]  

(6.4)

which defines fractional stationary of order \( \alpha \) and type \( \beta \). Of course, for \( \alpha = 1 \) this definition reduces to the conventional definition of stationary.

Equation (5.3) is solved by

\[ y(t) = \frac{y_0 d^{\alpha,\beta(\alpha - 1)}(1 - \beta)(\alpha - 1)}{\Gamma(1 - \beta)(\alpha - 1) + 1)} . \]

For more details, see [17]. Note, the fractional integral

\[ I_{0+}^{(1 - \beta)(\alpha - 1)} y(t) = y_0 , \]

remains conserved and constant, for all \( t \) while the function itself varies. In particular

\[ \lim_{t \to 0^+} y(t) = \infty \]

and

\[ \lim_{t \to \infty} y(t) = 0 . \]

For \( \beta = 1 \) and \( \alpha = 1 \), one recovers \( y(t) = y_0 \) as usual. This type of stationary states for which a fractional integral rather than the function itself is constant were first discussed in [4]. It seems to us that the lack of knowledge about fractional stationary is partially responsible for the difficulty of deciding which type of fractional derivative should be used.

In order to keep track of the past and future of the dynamics we need to consider the operator \( _a D_t^\alpha \) and \( _b D_t^\alpha \), and considering \( \alpha \neq \beta \) is only here for convenience. This can be used to take into account a different quantity information from the past and the future.

Let \( _a D_t^\alpha \) and \( _b D_t^\beta \) be given. We look for an operator \( D^{\alpha,\beta} \) of the form

\[ D^{\alpha,\beta} = M\left( _a D_t^\alpha , _b D_t^\beta \right) \]

(6.5)

Where \( M: \mathcal{R}^2 \to C \) is a mapping which does not depends on \( (\alpha, \beta) \), satisfying the following conditions:

1. If \( y(t) \in C^1 \) then when \( \alpha = \beta = m, m \in \mathbb{N}^* \), \( D^{m,m}_t y(t) = \frac{d^m y}{dt^m} \).
2. \( M \) is a \( \mathcal{R} \)-linear mapping, which is only the simplest dependence of the operator \( D \) with respect to \( _a D_t^\alpha \) and \( _b D_t^\beta \).
3. The mapping \( M \) is invertible, which means that the data of \( D^{\alpha,\beta} \) on a given function \( x \) at point \( t \) allows us to recover the left and right RL derivatives of \( x \) at \( t \), so information about \( x \) in a neighborhood of \( x(t) \).
4. Let \( y \in C^0 \) be a real valued function possessing left and right classical derivatives at point \( t \), denoted by \( \frac{d_{\pm} y}{dt} = -\frac{d_{\pm} x}{dt} \), then we impose that \( D^{\alpha,\beta}_t y(t) = \frac{d_{\pm} y}{dt} \).

Condition (4) must be seen as the non-differentiable pendant of condition (1). Indeed, condition (1) can be rephrased as follows: if \( x \in C^0 \) is such that \( d^+ y \) and \( d^- y \) exist and satisfy \( d^+ y = d^- y \) then \( D^{1,1}_t y = d^+ y = d^- y \).

**Theorem (5):** For all \( a, b \in \mathcal{R}, a < b \), the fractional operator of order \( (\alpha, \beta) \), \( \alpha > 0, \beta > 0 \), denoted by \( D^{\alpha,\beta}_\mu \), and satisfying conditions (1), (2), (3) and (4) are of the form (6.1).

**Proof:**
Let $M(x,y) = px + qy + i(rx + sy)$, with $p, q, r, s \in \mathbb{R}$. From condition (1), with $y = -x$ corresponding to the operator a choice of operator $\frac{d}{dx}, -\frac{d}{dx}$, we have $p-q = 1, r-s = 0$, and then already have an operator of the form

$$D^\alpha = \left[p_a D^\alpha + \frac{d}{dx}\right] \left[p_a + i\mu\right] D^\alpha + i D^\alpha$$

and the assumption (3) only impose that $q \neq 0$. Now, by condition (4), with $p+(p-1) = 0$ and $2q = 1$, so that $p = q = \frac{1}{2}$ we have

$$D_{a,b} = \frac{1}{2} \left[ a D^\alpha - D_b^\alpha \right] + i \frac{1}{2} \left[ a D^\alpha + D_b^\alpha \right],$$

and therefore, we have

$$D^\alpha = \frac{1}{2} \left[ a D^\alpha - D_b^\alpha \right] + i \mu \left[ a D^\alpha + D_b^\alpha \right].$$

When $\alpha = \beta = 1$, we obtain $D^\alpha = \frac{d}{dx}$, and the free parameter $\mu$ can be used to reduce the operator $D^\alpha$ to some special cases of importance. Let us denoted by $y(t)$ a given real valued function.

For $\mu = -i$ we have $D^\alpha = a D^\alpha$ then dealing with an operator using the future state denoted by $\mathcal{F}_a(x)$ of the underlying function, i.e. $\mathcal{F}_a(x) = \{y(s), s \in [a, b]\}$.

For $\mu = i$, we obtain $D^\alpha = b D^\alpha$ then dealing with an operator using the past state denoted by $\mathcal{G}_b(x)$ of the underlying function, i.e. $\mathcal{G}_b(x) = \{y(s), s \in [a, b]\}$.

When $a = -\infty$ and $b = \infty$, we denoted the associated operator $D^\alpha$ by $D^\alpha$ , i.e.

$$D^\alpha = \frac{1}{2} \left[ D^\alpha - D_b^\alpha \right] + i \mu \frac{1}{2} \left[ D^\alpha + D_b^\alpha \right],$$

where $\mu \in \mathbb{C}$. (6.6)

Now, we consider the following useful lemma, also see [4]:

**Lemma:** For all $f, g \in a D^\alpha$, we have

$$\int_a^b D^\alpha f(t) g(t) dt = -\int_a^b f(t) D^\alpha g(t) dt$$

provide that $f(a) = f(b) = 0$ or $g(a) = g(b) = 0$.

**Proof:**

We have

$$\int_a^b D^\alpha f(t) g(t) dt = \int_a^b f(t) \left[ (D^\alpha - a D^\alpha) + i \mu D^\alpha \right] g(t) dt$$

exchanging the role of $\alpha$ and $\beta$ in $\left[ (D^\alpha - a D^\alpha) + i \mu D^\alpha \right]$, we obtain the operator $\left[ (D^\alpha - a D^\alpha) + i \mu D^\alpha \right]$ which can be written as $-\left[ (D^\alpha - a D^\alpha) + i \mu D^\alpha \right] = -D^\alpha$. This concludes the proof.

Now, the classical product rule for Riemann-Liouville derivatives for all $\alpha > 0$, can be presented as in the following corollary:

**Corollary:**

$$\int_a^b D^\alpha f(t) g(t) dt = -\int_a^b f(t) D^\alpha g(t) dt$$

as long as $f(a) = f(b) = 0$ or $g(a) = g(b) = 0$.

This formula gives a strong connection between $a D^\alpha$ and $b D^\alpha$ via a generalized integration by part. This relation is responsible for the emergence of $b D^\alpha$ in problems of fractional calculus of variations only dealing with $a D^\alpha$. 

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As an application to such extended operator, is in finding the solution $f(.)$ to minimizes the following fractional problem:

$$V^{\alpha,\beta}_{\mu}(f) = \int_a^b L(D^{\alpha,\beta}_\mu f(s), g(s), s)(t-s)^{\alpha-1}ds$$

by considering a smooth manifold $M$, $L$ be a smooth Lagrangian function $L:C^r \times R^r \times R \rightarrow R$, $r \geq 1$, and any piecewise smooth path $f[a,b] \rightarrow M$, satisfying fixed boundary conditions $f(a)=f_a$ and $f(b)=f_b$. For more details see [1].

References


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