About Fractional Operators: Behavior and Extension

Alauldin Noori Ahmed , Ahmed Ayyoub Yousif

Al-Nahrain University - College of Science - Department of Mathematic & Computer Applications – Iraq

Receiving Date: 25-03-2010 **- Accept Date:** 16-01-2011

Abstract

In this paper, we are study some behavior fractional operators solutions, such as existence, uniqueness, stabilities and extended formulas for a dynamic multi-fractional order differential equations. Theorems are stated and proved based on the definitions of Riemann-Liouville fractional integral and Grunwald-Letnikov fractional derivative. Also, a simple numerical approach has been implemented to solve such system.

Key Words: Fractional Calculus, Existence and Uniqueness solution, Stability.

1. Introduction

As to the history of fractional calculus [10], already in 1695 L'Hospital raised the question as to the meaning of $\frac{d^n y}{dx^n}$ if n=½, that is "what if n is fractional". "This is an apparent paradox from which, one day, useful consequences will be drawn", Leibniz replied, together with "$d^{\frac{1}{2}}x$ will be equal to $\sqrt{dx:x}$". Lacroix, was the first to mention in some two pages a derivative of arbitrary order in a 700 page text book of 1819. Although fractional derivatives have a long mathematical history, for many years they were not used in many different sciences, but in recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. Recently, there has been some attempt to solve linear problems with multiple fractional derivatives problems, see [2]. Not much has been done for the nonlinear problems. Problems of stability appeared for the first time in mechanics during the investigation of an equilibrium state of a system. A simple reflection may show that some equilibrium state of a system is stable with respect to small perturbations. The existing methods developed so far for stability check are mainly for integer order systems. However, for the fractional order systems, it is difficult to evaluate the stability by simple examining its characteristic equation either by finding it's dominate roots or by using other algebraic methods. Direct check of the stability of fractional order system using polynomial criteria is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudo-polynomial function of fractional powers of the complex variables, see [2].The study of stability of such systems focuses a great interest. We can cite in this domain, the works in [12] and [13] for the stability of linear fractional systems, while the works in [9], and [11] are for the stability of fractional systems with time delay.

This paper, is organized in the following, as well as the introduction in section one. The preliminaries and notations of the problem study is presented in section two, by recalling the definitions of the left and right fractional derivatives of Riemann-Liouville and Gunwald-Letnikov fractional derivatives. The existence and uniqueness solution theorem is stated and proved in section three. Theorems for the stability solution on finite and infinite time interval are stated and proved, in section four. In section five, other stability theory is presented for different structure fractional systems. In section six, the left and right fractional derivatives operator $D^{\alpha,\beta}$, which have satisfy a semi-group property and the adequate functional spaces on which, a previous work of Erwin and Roop [2], can be extended, to discuss some properties and cases, such as fractional stationary in the fractional order dynamic systems, to keep its track information of the past and future.

In this paper, we will consider the system of nonlinear multi-fractional order differential equations

$$y_i^{\alpha_i}(t) = f_i(t, Y(t)), \qquad i=1,\ldots,m \tag{1.1}$$

where $Y(t) = (y_1(t),\ldots, y_m(t))^T$ is the solution of the system (1), and $0<\alpha_i\leq 1$, in which two main definitions of *Riemann-Liouville fractional integration* , and *Grunwald-Letnikov fractional derivative* are used.

2. Definitions

([8], [9], [10], and [11]).

Definition 1: (Left and Right Riemann-Liouville Fractional integral) Let y be a function defined on (a, b), and $\alpha > 0$. Then the left and right Riemann-Liouville fractional integral of order α are defined to be

$$_{\alpha}D_t^{-\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)ds,$$

$$_{t}D_b^{-\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} y(s)ds \tag{2.1}$$

Left and right (RL) integrals satisfy some important properties like the semi-group property. For more details, we refer to [3].

Definition 2: (Left and Right Riemann-Liouville fractional derivative) Let $\alpha > 0$, the left and right Riemann-Liouville derivative of order α, denoted by $_{\alpha}D_t^{\alpha}$ and $_{t}D_b^{\alpha}$ respectively, are defined by

$$_{\alpha}D_t^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} y(s)ds,$$

$$_{t}D_b^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (t-s)^{n-\alpha-1} y(s)ds \tag{2.2}$$

where $n=[\alpha]+1$, such that $n-1\leq\alpha<n$.

The traditional definition of the derivative of a continuous function is expressed in terms of a limiting procedure. Consider the Taylor expansion of the continuous function $Y(t)$ given by

$$Y(t+\delta)=Y(t)+\delta DY(t)+\frac{\delta^2}{2!}D^2Y(t)+\cdots+\frac{\delta^n}{n!}D^nY(t)+\cdots$$

where D is the derivative operator and writing the upper shift operator, E_δ, as $E_\delta Y(t)=Y(t+\delta)$, we obtain $E_\delta Y(t)=e^{\delta D}Y(t)$.
Therefore, we have $E_\delta=e^{\delta D}$, and we can write
$E_\delta=1-\Delta_-= e^{\delta D}$,
so that the right-difference operator becomes

$$\Delta_- =1-E_\delta=1-e^{\delta D} . \qquad (2.3)$$

In this way we have the definition of the derivative operator

$$\lim_{\delta\to 0}\frac{\Delta_-}{\delta}=-D$$

it is clear from the above, that if t is the time derivative operator and when applied to a continuous function of time yields

$$\frac{d}{dx}Y(t)=-\lim_{\delta\to 0}\frac{\Delta_- Y(t)}{\delta}, \qquad (2.4)$$

we can do a similar construction for the downshift operator $E_\delta^{-1}Y(t)=Y(t-\delta)$, yields

$$\lim_{\delta\to 0}\frac{\Delta_+}{\delta}=D \qquad (2.5)$$

So (2.4) and (2.5) are equivalent mathematical expressions of the derivative operator. To be on safe side, we write the derivative operator in the symmetric form

$$\lim_{\delta\to 0}\frac{\Delta_+-\Delta_-}{2\delta}=D$$

Now, we extend the definition of integer-order derivatives to that of fractional order by generalizing the definition of the limit of finite differences. Consider (2.3) raised to the α^{th} power

$$\Delta_\pm^\alpha=(1-e^{-\delta\frac{d}{dt}})^\alpha$$

so that we can write for the left-side fractional time derivative

$$\lim_{\delta\to 0}\frac{\Delta_+^\alpha}{\delta^\alpha}Y(t)=\lim_{\delta\to 0}\frac{(1-e^{-\delta\frac{d}{dt}})^\alpha}{\delta^\alpha}Y(t)=\frac{d^\alpha}{dt^\alpha}Y(t), \qquad (2.6)$$

where we have replaced D with the time-derivative operator. In a similar way we can write

$$\lim_{\delta\to 0}\frac{\Delta_-^\alpha}{\delta^\alpha}Y(t)=\lim_{\delta\to 0}\frac{(1-e^{-\delta\frac{d}{dt}})^\alpha}{\delta^\alpha}Y(t)=\left(-\frac{d^\alpha}{dt^\alpha}\right)^\alpha Y(t), \qquad (2.7)$$

for the right-side fractional derivative. We can use (2.6) and (2.7), to write the more general formal expression for the left-side and right-side fractional derivative operators as

$$D_{\pm}^{\alpha}Y(t) = \lim_{\delta\to 0}\frac{\Delta_{\pm}^{\alpha}Y(t)}{\delta^{\alpha}} \tag{2.8}$$

which is mentioned in [10], as the Grunwall-Letnikov fractional derivative

$$\Delta_{\delta}^{\alpha}y(x) = \sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j}y(x - j\delta) \tag{2.9}$$

which is coincides with the classical r^{th} right handed difference $\Delta_{\delta}^{\alpha}y(x)$ for $\alpha = r \in N$, (Note that $\binom{r}{j} = 0$ for $j \geq r+1$). The existence of $D^{\alpha}y$ in the uniform norm is equivalent to the fact that the r^{th} ordinary derivative $y^{(r)}(x)$ exists for all x and is continuous, and this suggests that $D^{\alpha}y$ can be handled by considering the limit in norm. Thus, $D^{\alpha}y$ will be function g in $C_{2\pi}$ or $L_{2\pi}^{p}$, $1 \leq p < \infty$, respectively, for which

$$\lim_{\delta\to 0}\left\| h^{-\alpha}\Delta_{\delta}^{\alpha}y(x) - g\right\| = 0$$

exists. Therefore, we can write the fractional derivative in the Grunwall-Letnikov sense as

$$D^{\alpha}y(t) = \lim_{\delta\to 0}(\delta)^{-\alpha}\sum_{j=0}^{\left[\frac{t}{\delta}\right]}(-1)^{j}\binom{\alpha}{j}y(t - j\delta) = \sum_{j=0}^{\left[\frac{t}{\delta}\right]}C_{j}^{\alpha}y(t - j\delta) \tag{2.10}$$

where $[t]$ is the integer part of t, δ is the step size, and

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha+1-j)} = \frac{\alpha!}{j!(\alpha-j)!}.$$

The definition of operator in the Grunwald-Letnikov sense (2.10) is equivalent to the definition of operator in the Riemann-Liouville sense (2.1). Nevertheless, the Grunwald-letnikov operator is more flexible and most straightforward in numerical calculations, in which, the solution of (2.10) is given by the following recurrence equation: (see [22])

$$y_0 = \beta,\ y_n = f(t, y_n) - \sum_{j=1}^{n}c_{j}^{\alpha}y_{n-j},\ (n = 1,2,3,\ \ldots), \tag{2.11}$$

where $t_n = n\delta$, $y_n = y(t_n)$ are positive functions for all n, and c_j^{α} are the Grunwald-letnikov coefficients defined as:

$c_0^{\alpha} = \delta^{-\alpha}$, $c_j^{\alpha} = \left(1 - \frac{1+\alpha}{j}\right)c_{j-1}^{\alpha}$, ($j$=1,2,3, …) for more details, see [9].

3. Existence and Uniqueness Solution:

By Approximating the fractional derivative in (1) by (2.10), leads to the numerical solution algorithm described by (2.11), in the following recursive relations [10]:

$$y_{i,0} = \beta,\ y_{i,n} = f_{i,n} - \sum_{j=1}^{l}c_{j}^{\alpha_i}y_{i,n-j},\ (n = 1,2,\ \ldots), \tag{3.1}$$

this algorithm is simple for computational performance for all values of α_i, $0<\alpha_i \leq 1$.
For details about the fractional difference method and its applications for solving fractional differential equations, we refer the reader to [11].

Set $\hat{D} = I \times C^*(t)$ where $C^*(t)$ is the class of all continuous column vectors $Y(t)$ with the norm

$$\|Y(t)\| = \sum_{i=1}^{m} \|y_i(t)\| = \sum_{i=1}^{m} \max_{i\in I} |y_i(t)|$$

Now, we can state the following theorem:

Theorem (1): Let $F(t,Y(t)) \in C^*(\hat{D})$, where $F(t, Y(t))=(f_1(t, Y(t)),\ldots, f_m(t, Y(t)))^T$, i.e. $f_i(t,Y(t)) \in C(\hat{D})$ for all $i=1,\ldots, m$, and each satisfies the Lipschtiz condition

$$|f_i(t,Y(t)) - f_i(t,X(t))| \leq k\sum_{l=0}^{m} |y_l(t) - x_l(t)| \text{ for all } i, \tag{3.2}$$

$(t,Y(t)),(t,X(t)) \in \hat{D}$ and $k=min\ k_i>0$.

If $\max C_j^{\alpha_i} \leq (1-k)$, (3.3)

then (1) has one and only one solution $Y(t)\in C(I)$ that satisfies $D^{\alpha_i}Y(t) \in C(I)$.

Proof:

From (2.11), If we write $TY_n = F(t,Y_n(t)) - \sum_{j=1}^{l} C_j^{\alpha_i} Y_{n-j}$, then for $(t,Y(t)),(t,X(t)) \in \hat{D}$, we get

$$\begin{aligned}
\|TY_n - TX_n\| &= \left\| \left(F(t,Y_n(t)) - \sum_{j=1}^{l} C_j^{\alpha_i} Y_{n-j} \right) - \left(F(t,X_n(t)) - \sum_{j=1}^{l} C_j^{\alpha_i} X_{n-j} \right) \right\| \\
&\leq \left\| (F(t,Y_n(t)) - F(t,X_n(t))) + \left(\sum_{j=1}^{l} C_j^{\alpha_i} Y_{n-j} - \sum_{j=1}^{l} C_j^{\alpha_i} X_{n-j} \right) \right\| \\
&\leq k\sum_{l=0}^{m} |y_l(t) - x_l(t)| + \sum_{j=1}^{l} \max |C_j^{\alpha_i}| \|Y_{n-j} - X_{n-j}\| \\
&\leq k\|Y_n - X_n\| + (1-k)\|Y_{n-j} - X_{n-j}\| \\
&= \|Y_n - X_n\|, \text{ as } n\longrightarrow\infty,\ Y_{n-j}\sim Y_n \text{ and } X_{n-j}\sim X_n.
\end{aligned}$$

Hence, the mapping $T: C^*(\hat{D}) \to C^*(\hat{D})$ is a contraction mapping, and then it has a fixed point $Y(t)=T(Y(t))$.

Providing the condition (3.3) and hence, there exists a unique solution $Y(t) \in C^*(\hat{D})$ for the system (1).■

4. Stability Solution

First, we set $I=[0,t_f]$, say, t_f is a finite suitable positive number, and define the norm

$$ {}_{0}^{t_f}\|Y(t)\| = \sum_{i=1}^{m} {}_{0}^{t_f}\|y_i(t)\| \quad (4.1)$$

where

$$ {}_{0}^{t_f}\|y_i(t)\| = \int_0^{t_f} |y_i(t)| dt \quad (4.2)$$

Riemann-Liouville fractional integration of order α_i is defined as:

$$ y_i(t) = I^{\alpha_i} f_i(t, Y(t)) = \frac{y_0 (t-t_0)^{\alpha_i}}{\Gamma(\alpha_i)} + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} f(s, Y(s)) ds \quad (4.3)$$

$= \frac{t^{\alpha_i - 1}}{\Gamma(\alpha_i)} \otimes f(t, Y(t))$, $\alpha_i > 0$ and $t > 0$.

In order to study the L_1- stability solution of the system (1), we must prove that each $y_i(t)$, (i=1,…,m) given by (4.3) is bounded, i.e. $\|y_i(t)\| < \infty$, for all i=1,…,m.

We define the convolution operator of f and g as follows:

$$ f \otimes g(t) = \int_0^t f(t-s) g(s) ds \quad (4.4)$$

therefore, the solution $y_i(t)$ can be rewritten using the product operator ⊗ as follows

$$ y_i(t) = (k_{\alpha_i} \otimes f_i)(t) \quad (4.5)$$

Where k_{α_i} with $0 < \alpha_i \leq 1$ is so called convolution kernel defined by

$k_{\alpha_i} = \frac{t^{\alpha_i - 1}}{\Gamma(\alpha_i)}$, for all i=1,2,…, m. (4.6)

Theorem (2): Let $k_{\alpha_i}, f_i \in C(R^+, R^+) \cap L_1([0, t_f])$, with $t_f > 0$ and finite, then the solution of (1) is stable, providing that

$\left| \frac{t^{\alpha_i}}{\alpha_i \Gamma(\alpha_i)} \right| < \varepsilon$, for all i=1,…,m. (4.7)

Proof:

With no loss of generality we can take y_0=0 and t_0=0. This reduces (4.3) into

$$ y_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} f(s, Y(s)) ds$$

First, we prove that ${}_{0}^{t_f}\|Y(t)\| < \infty$.

We have

$$ {}_{0}^{t_f}\|Y(t)\| = \sum_{i=1}^{m} {}_{0}^{t_f}\|y_i(t)\| = \sum_{i=1}^{m} \left(\int_0^{t_f} (k_{\alpha_i} \otimes f_i)(t) dt \right)$$

So, we need only to prove that ${}_{0}^{t_f}\|y_i(t)\| < \infty$

$$ {}_{0}^{t_f}\|y_i(t)\| = \int_0^{t_f} k_{\alpha_i} \otimes f_i(t, Y(t)) dt$$

$= \int_0^{t_f} \int_0^{t_f} k_{\alpha_i}(t-s) f_i(s, Y(s)) ds dt$ (by Fubini Theorem for positive functions)

$$= \int_0^{t_f} f_i(s, Y(s)) \left(\int_0^{t_f - s} k_{\alpha_i}(t-s) d(t-s) \right) ds \text{ (by using the change of variables)}$$

$$= \int_0^{t_f} f_i(s, Y(s)) \left(\int_0^{t_f - s} k_{\alpha_i}(t) dt \right) ds ,$$

since all k_{α_i} is positive, we have $\int_0^{t_f - s} k_{\alpha_i}(t)dt \le \int_0^{t_f} k_{\alpha_i}(t)dt$

and ${}_0^{t_f}\|y_i(t)\| \le \left(\int_0^{\infty} k_{\alpha_i}(t)dt \right)\left(\int_0^{\infty} f_i(s, Y(s))ds \right)$

since t_f is finite, then the right side of the inequality is converge, and we get

$$ {}_0^{t_f}\|y_i(t)\| \le \left(\frac{(t_f)^{\alpha_i}}{\alpha_i \Gamma(\alpha_i)} \right) \int_0^{\infty} f_i(s, Y(s)) ds < \infty \tag{4.8}$$

provided that, the condition (4.7) is satisfied.

Second, if we consider two solutions $Y(t)$ and $X(t)$ with different initial points Y_0 and X_0 with $|Y_0 - X_0| < \delta(t_f)$. Perform the same above steps for ${}_0^{t_f}\|Y(t) - X(t)\|$, we get

$$\begin{aligned} {}_0^{t_f}\|Y(t) - X(t)\| &= \sum_{i=1}^{m} {}_0^{t_f}\|y_i(t) - x_i(t)\| = \sum_{i=1}^{m} \int_0^{t_f} |y_i(t) - x_i(t)| dt \\ &= \sum_{i=1}^{m} \int_0^{t_f} \left| \left\{ \tfrac{(y_0 - x_0)}{\Gamma(\alpha_i)} t^{\alpha_i - 1} + \left(\int_0^{t_f} k_{\alpha_i} \otimes (f_i(s, Y(t) - f_i(s, X(t)) dt \right) \right\} \right| dt \qquad (4.9) \\ &\le \sum_{i=1}^{m} \int_0^{t_f} \left\{ \left| \tfrac{(y_0 - x_0)}{\Gamma(\alpha_i)} t^{\alpha_i - 1} \right| + \left(\int_0^{t_f} \left| k_{\alpha_i} \otimes (f_i(s, Y(t) - f_i(s, X(t)) \right| dt \right) \right\} dt \end{aligned}$$

As we have seen from (4.8), the second term of (4.9) is converge and bounded. Also the first term of (4.9) is converge and bounded, providing that condition (4.7) is satisfied. Therefore we get

$${}_0^{t_f}\|Y(t) - X(t)\| < \infty . \blacksquare$$

The generalization to the infinite case where $t \in R^+$ is not possible because the kernel $k_{\alpha_i}(t)$ does not belong to $L_1(R^+)$, and then $\int_0^{\infty} k_{\alpha_i} \otimes f_i(t, Y(t))dt$ does not converge, and $\int_0^{\infty} k_{\alpha_i} \otimes f_i(t, Y(t))dt \le \left(\int_0^{\infty} f_i(s, Y(s))dt \right)\left(\int_0^{\infty} k_{\alpha_i}(t)dt \right)$ does not true.

Also, even if the system is defined since $t_0>0$, the generalization to the infinite case where $t \in [\, t_0, \infty)$ is not possible, since the solution of the system (1) given by (4.3) can't be defined by using a convolution product which is commutative. However, our kernel function given by (4.6) does not depend on the initial time. Therefore, in order to overcome the problems difficulties, we consider our attention to the formal derivation of fractional derivatives using the continuum limit of finite difference equations.

The definition of operator in the Grunwald-Letnikov sense (2.10) is equivalent to the definition of operator in the Riemann-Liouville sense (2.2). Nevertheless, the Grunwald-letnikov operator is more flexible and most straightforward in numerical calculations, (see [22]).

If we denote $y_i(t)=\lim_{n\to\infty} y_n$, for all i=1,…,m, and $f(t,Y(t))=\lim_{n\to\infty} f(t,Y(t_n))$.

Define the norm of $Y(t)$ as following

$$_0^\infty\|Y(t)\|=\sum_{i=1}^{m} {}_0^\infty\|y_i(t)\| \tag{4.10}$$

Theorem (3): The solution of (1) is asymptotically stable, providing that $f_i \in L^p[0,\infty)$, with $p\geq 1$, (for all i=1,…,m).

Proof:

Consider the solution (3.1).

First, we will prove that the system (1) is stable.

$$\|y_i(t)\|=\left\|f_i(t,Y(t))-\sum_{j=1}^{n} c_j^{\alpha_i} y_i(t_{n-j})\right\|$$

Since $0<c_j^{\alpha_i}<1$, for all j=1,2,…,n, and all $0\leq y_i(t_{n-j})\leq\beta$, then all $\lim_{n\to\infty}\sum_{j=1}^{n} c_j^{\alpha_i} y_i(t_{n-j})$ is converge and bounded, and since $f_i \in L^p(0,\infty)$, with $p\geq 1$, (for all i=1,…,m), the right side is bounded, and we have $\|y_i(t)\|<\infty$, and then $\|Y(t)\|<\infty$.

Second, if we consider two solutions $Y(t)$ and $X(t)$ with two different initial values Y_0 and X_0, with $|y_i(t_0)-x_i(t_0)|<\epsilon$, for all i=1,…,m, and $0<\epsilon<1$, then $\lim_{n\to\infty}\left(y_i(t_{n-j})-x_i(t_{n-j})\right)\to 0$ and $\lim_{n\to\infty}\left(f_i(t,Y(t_n))-f_i(t,X(t_n))\right)\to 0$ as $n\to\infty$.

Performing same as above steps, we get $\|Y(t)-X(t)\|\to 0$, as $t\to\infty$.

Therefore, the solution of the system (1) is asymptotically stable.■

5. Other Stability Approach:

([12], [17], [20], [23] and [24])

As we mentioned before in this paper, exponential stability cannot be used to characterize asymptotic stability of fractional order systems. Many general fractional order systems can be described (or transform) in the following form

$$G(s)=\frac{b_m s^{\beta_m}+\cdots+b_1 s^{\beta_1}+b_0 s^{\beta_0}}{a_m s^{\alpha_m}+\cdots+a_1 s^{\alpha_1}+a_0 s^{\alpha_0}}=\frac{Q(s^{\beta_k})}{P(s^{\alpha_k})}, \tag{5.1}$$

where a_k (k=1,…,m), b_k (k=1,…,n) are constant; and α_k (k=1,…,n), β_k (k=1,…,m) are arbitrary real numbers and without loss of generality they can be arranged as $\alpha_n>\alpha_{n-1}>\cdots>\alpha_0$ and $\beta_m>\beta_{m-1}>\cdots>\beta_0$. Also, the form (5.1) can be put into the following form

$$G(s) = K_0 \frac{\sum_{k=0}^{N} b_k (s^{\beta})^k}{\sum_{k=0}^{N} a_k (s^{\alpha})^k} = \frac{Q(s^{\beta})}{P(s^{\alpha})} = K_0 \left[\sum_{i=1}^{N} \frac{A_i}{s^{\alpha} + \lambda_i} \right] \tag{5.2}$$

with $a_k=ak$, $\beta_k=\alpha k$, $(0<\alpha<1)$. $\forall k\in Z$, $N>M$ and λ_i (i=1,…,N) are the roots of pseudo-polynomial $P(s^{\alpha})$ or the system poles which are assumed to be simple without loss of generality. The analytical solution of the system (5.2) can be expressed as

$$y(t) = L^{-1}\left\{ K_0 \left[\sum_{i=1}^{N} \frac{A_i}{s^{\alpha} + \lambda_i} \right] \right\} = K_0 \sum_{i=1}^{N} A_i t^{\alpha} E_{\alpha,\alpha}(-\lambda_i t^{\alpha}). \tag{5.3}$$

where $E_{\alpha,\alpha}(.)$ is the Mittag-Leffler function of two parameters [10].

Several authors using several methods that a geometrical method of complex analysis based on the argument principle of the roots of the characteristic equation can be used for the stability check. The stability condition can be stated as follows

Theorem (4): The system (5.2) is stable if and only if $|\arg(\lambda_i)|>\alpha \frac{\pi}{2}$, for all i. with λ_i is the i^{th} root of $P(s^{\alpha})$.

A new definition was introduced in [20], can be presented as follows:

Definition 4: The trajectory $y(t)$=0 of the system (1) t^{-r} is asymptotically stable if there is a positive real r such that: $\forall\|y(t)\|$ with $t\le t_0$, $\exists N(y(t))$, such that $\forall t\ge t_0$, $\forall\|y(t)\|\le N t^{-r}$.

The fact that the components of $y(t)$ slowly decay towards 0 following t^{-r} leads to fractional systems sometimes being called long memory systems. Power law stability t^{-r} is a special case of the Mittage-Leffler stability [18]. According to the stability theorem defined in [24], the equilibrium points are asymptotically stable for $r_1= r_2=\cdots=r_n=r$ if all the eigenvalues λ_i (i=1,…,n) of the Jacobian matrix $J=\frac{\partial f}{\partial y}$ evaluated at the equilibrium, satisfy the condition $|\arg(\lambda_i)|>r\frac{\pi}{2}$, i=1,…,n [23].

If $r_1\neq r_2\neq\cdots\neq r_n$ and suppose that m is the LCM of the denominators u_i's of r_i's, where $r_i=\frac{v_i}{u_i}, v_i, u_i \in Z^+$ for i=1,…,n and setting $\gamma=\frac{1}{m}$. System (1) is asymptotically stable if $|\arg(\lambda)|>\gamma\frac{\pi}{2}$, for all roots λ of the equation

$$\det(diag([\lambda^{mr_1}\lambda^{mr_2}\ldots\lambda^{mr_{n1}}])-J)=0. \tag{5.4}$$

A necessary stability condition for the system (1) to remain chaotic is keeping at least one eigenvalue λ in the unstable region [23]. The above theory can demonstrated by the following nonlinear fractional order system:

$$D_t^{0.8} y_1(t) = 35[y_2(t) - y_1(t)]$$
$$D_t^{1.0} y_2(t) = -7y_1(t) - y_1(t)y_3(t) + 28y_2(t)$$
$$D_t^{0.9} y_3(t) = y_1(t)y_2(t) - 3y_3(t)$$

The system has three equilibrium at (0,0,0), (7.94,7.94,21), and (−7.94,−7.94,21). The Jacobian matrix of the system evaluated at (y_1^*, y_2^*, y_3^*) is:

$$\begin{bmatrix} -35 & 35 & 0 \\ -7-y_3^* & 28 & -y_1^* \\ y_2^* & y_1^* & -3 \end{bmatrix},$$

The two last equilibrium points are saddle points and (5.4) becomes as follows:

$\lambda^{27}+35\lambda^{19}+3\lambda^{18}-28\lambda^{17}+105\lambda^{10}-21\lambda^{8}+4410=0$

And this characteristic equation has unstable roots $\lambda_{1,2}=1.2928\pm0.2032j$, $|\arg(\lambda_{1,2})|=0.1560$, and therefore, the system satisfy the stability condition.

6. Extended Fractional Operator [4]:

Looking for the past P_t and the future F_t of a given dynamical process $x(s)$, $s\in\Re$ at time t, *i.e.* on the information $P_t=\{x(s),a\leq s\leq t\}$ and $F_t=\{x(s),t\leq s\leq b\}$ where a and b can be chosen and depends on the amount of information we are keeping from the past and the future. This induces the fact that we look for two quantities, not yet define that we denote by $d_-x(t)$ and $d_+x(t)$ from the point of view of derivatives. The past and future information can be weighted, achieved by using a weight $\frac{1}{|t-s|^{\alpha+1}}$ and regularizing the corresponding function. We then are lead to two quantities $d_-^\alpha x(t)$ and $d_+^\alpha x(t)$, which represent weighted information on the past and future behavior of the dynamical process.

The previous idea is well formalized by the left and right (Riemann-Liouville) derivatives [18]. We have used in this paper the left and right Riemann-Liouville derivatives with different indices for the left and right differentiation, *i.e.* we consider ${}_\alpha D_t^\alpha$ and ${}_t D_b^\alpha$. The extended operator depends naturally on these two operators and is denoted $D^{\alpha,\beta}$. However, this operator does not reduce to the ordinary derivative only when α=β=1. In [1] Agrawal has studied fractional variation problems using the Riemann-Liouville derivatives. He notes that even if the initial functional problems only deals with the left Riemann-Liouville derivative, the right Riemann-Liouville derivative appears naturally during the computations. Several useful functional spaces were introduced. Let $I\subset R$ be an open interval (which may be unbounded). We denote by $C_0^\infty(I)$ the set of all functions $x\in C^\infty(I)$ that vanish outside a compact subset K of I. In the classical integrals, and derivatives of order one, are obtained by setting α=1. In this section, the fractional derivative operator of order (α, β) introduced in [4], is presented in the following form:

$$D_\mu^{\alpha,\beta}=\tfrac{1}{2}\left[{}_\alpha D_t^\alpha-{}_t D_b^\beta\right]+i\mu\left[{}_\alpha D_t^\alpha-{}_t D_b^\beta\right] \qquad (6.1)$$

where a, $b\in$ R, $a<b$, $\mu\in$ C and α, $\beta>0$,and its generalization present in the following generalized definition, based on differentiating fractional integrals.

Definition 5: The (right-left-sided) fractional derivative of order 0<α<1 and type 0≤β≤1 with respect to t is defined by

$$D_{a\pm}^{\alpha,\beta}y(t)=\left[\pm I_{a+}^{(1-\beta)(1-\alpha)}\tfrac{d}{dt}\left(I_{a+}^{(1-\beta)(1-\alpha)}y\right)\right] \qquad (6.2)$$

for functions for which the expression on the right hand side exists.

The above definition is equivalent to definition of right and left Riemann-Liouville fractional derivative, when β=0 and 1, which seems that the fractional derivatives of general type $0<\beta<1$ have not been considered. To explain the implementation of such operator, it can be possible to discuss the infinitesimal form of *fractional stationary*, by considering the following equation

$$D_{a\pm}^{\alpha,\beta} y(t) = 0 \qquad (6.3)$$

with initial condition

$$I_{0+}^{(1-\beta)(1-\alpha)} y(0+) = y_0 \qquad (6.4)$$

which defines fractional stationary of order α and type β. of course, for α=1 this definition reduces to the conventional definition of stationary.

Equation (5.3) is solved by $y(t) = \dfrac{y_0 t^{(1-\beta)(\alpha-1)}}{\Gamma((1-\beta)(\alpha-1)+1)}$.

For more details, see [17]. Note, the fractional integral $I_{0+}^{(1-\beta)(1-\alpha)} y(t) = y_0$, remains conserved and constant, for all t while the function itself varies. In particular $\lim_{t\to 0} y(t) = \infty$ and $\lim_{t\to\infty} y(t) = 0$. For β=1 and α=1, one recovers $y(t)=y_o$ as usual. This type of stationary states for which a fractional integral rather than the function itself is constant were first discussed in [4]. It seems to us that the lack of knowledge about fractional stationary is partially responsible for the difficulty of deciding which type of fractional derivative should be used.

In order to keep track of the past and future of the dynamics we need to consider the operator ${}_{\alpha}D_t^{\alpha}$ and ${}_{t}D_b^{\beta}$, and considering α≠β is only here for convenience. This can be used to take into account a different quantity information from the past and the future.

Let ${}_{\alpha}D_t^{\alpha}$ and ${}_{t}D_b^{\beta}$ be given. We look for an operator $D^{\alpha,\beta}$ of the form

$$D^{\alpha,\beta} = M\left({}_{\alpha}D_t^{\alpha}, {}_{t}D_b^{\beta}\right) \qquad (6.5)$$

Where M: $\Re^2 \to C$ is a mapping which does not depends on (*α, β*), satisfying the following conditions:

1. If $y(t) \in C^1$ then when α=β=*m*, *m*∈ℵ*, $D^{m,m} y(t) = \frac{d^m y}{dt^m}$.
2. *M* is a ℜ-linear mapping, which is only the *simplest dependence* of the operator *D* with respect to ${}_{\alpha}D_t^{\alpha}$ and ${}_{t}D_b^{\beta}$.
3. The mapping *M* is invertible, which means that the data of $D^{\alpha,\beta}$ on a given function *x* at point *t* allows us to recover the left and right RL derivatives of *x* at *t*, so information about *x* in a neighborhood of *x*(*t*).
4. Let $y \in C^0$ be a real valued function possessing left and right classical derivatives at point *t,* denoted by $\frac{d_+ y}{dt} = -\frac{d_- y}{dt}$, then we impose that $D_{1,1} y(t) = i \frac{d_+ y}{dt}$.

Condition (4) must be seen as the non-differentiable pendent of condition (1). Indeed, condition (1) can be rephrased as follows: if $x \in C^0$ is such that $d^+ y$ and $d^- y$ exist and satisfy $d^+ y = d^- y$ then $D^{1,1} y = d^+ y$.

Theorem (5): For all *a*, *b* ∈ℜ, a < b, the fractional operator of order (α, β), α > 0, β>0, denoted by $D_{\mu}^{\alpha,\beta}$, and satisfying conditions (1), (2), (3) and (4) are of the form (6.1).

Proof :

Let $M(x,y) = px + qy + i(rx + sy)$, with $p, q, r, s \in \Re$. From condition (1), with $y=-x$ corresponding to the operator a choice of operator $\frac{d}{dt}, -\frac{d}{dt}$, we have $p-q=1$, $r-s=0$, and then already have an operator of the form

$D^{\alpha,\beta} = \left[p\,{}_{\alpha}D_t^{\alpha}(p-1)\,{}_tD_b^{\beta}\right] + iq\left[{}_{\alpha}D_t^{\alpha} + {}_tD_b^{\beta}\right]$

and the assumption (3) only impose that $q \neq 0$. Now, by condition (4), with $p+(p-1)=0$ and $2q=1$, so that $p=q=½$ we have

$D_{\alpha,\beta} = \frac{1}{2}\left[{}_{\alpha}D_t^{\alpha} - {}_tD_b^{\beta}\right] + i\frac{1}{2}\left[{}_{\alpha}D_t^{\alpha} + {}_tD_b^{\beta}\right]$,

and therefore, we have

$D_{\mu}^{\alpha,\beta} = \frac{1}{2}\left[{}_{\alpha}D_t^{\alpha} - {}_tD_b^{\beta}\right] + i\mu\left[{}_{\alpha}D_t^{\alpha} - {}_tD_b^{\beta}\right]$.■

When α=β=1, we obtain $D_{\mu}^{\alpha,\beta} = \frac{d}{dt}$, and the free parameter μ can be used to reduce the operator $D_{\mu}^{\alpha,\beta}$ to some special cases of importance. Let us denoted by $y(t)$ a given real valued function.

For $\mu=-i$ we have $D_{\mu}^{\alpha,\beta} = {}_{\alpha}D_t^{\alpha}$ then dealing with an operator using the *future state* denoted by $\mathcal{F}_t(x)$ of the underlying function, *i.e.* $\mathcal{F}_t(x)=\{y(s), s\in[a,t[\}$.

For $\mu=i$, we obtain $D_{\mu}^{\alpha,\beta} = -{}_tD_b^{\beta}$ then dealing with an operator using the *past state* denoted by $\wp_t(x)$ of the underlying function, *i.e.* $\wp_t(x)=\{y(s), s\in[t,b[\}$.

When $a=-\infty$ and $b=\infty$, we denoted the associated operator $D_{\mu}^{\alpha,\beta}$ by $D_{\mu}^{\alpha,\beta}$, *i.e.*

$D_{\mu}^{\alpha,\beta} = \frac{1}{2}\left[D^{\alpha} - D_*^{\beta}\right] + i\mu\frac{1}{2}\left[D^{\alpha} - D_*^{\beta}\right]$, where $\mu \in C$. (6.6)

Now, we consider the following useful lemma, also see [4]:

Lemma: For all $f, g \in {}_a^{\alpha}D_b^{\beta}$, we have

$$\int_a^b D_{\mu}^{\alpha,\beta} f(t)g(t)dt = -\int_a^b f(t)D_{-\mu}^{\alpha,\beta} g(t)dt \qquad (6.7)$$

provide that $f(a)=f(b)=0$ or $g(a)=g(b)=0$.

Proof:

We have $\int_a^b D_{\mu}^{\alpha,\beta} f(t)g(t)dt = \int_a^b f(t)\left[\left({}_tD_b^{\alpha} - {}_{\alpha}D_t^{\beta}\right) + i\mu\left({}_tD_b^{\alpha} + {}_{\alpha}D_t^{\beta}\right)\right]g(t)dt$

exchanging the role of α and β in $\left({}_tD_b^{\alpha} - {}_{\alpha}D_t^{\beta}\right) + i\mu\left({}_tD_b^{\alpha} + {}_{\alpha}D_t^{\beta}\right)$, we obtain the operator $\left({}_tD_b^{\beta} - {}_{\alpha}D_t^{\alpha}\right) + i\mu\left({}_tD_b^{\beta} + {}_{\alpha}D_t^{\alpha}\right)$ which can be written as $-\left({}_tD_b^{\beta} - {}_{\alpha}D_t^{\alpha}\right) + i\mu\left({}_tD_b^{\beta} + {}_{\alpha}D_t^{\alpha}\right) = -D_{-\mu}^{\alpha,\beta}$

This concludes the proof.■

Now, the *classical product rule* for Riemann-Liouville derivatives for all α > 0, can be presented as in the following corollary:

Corollary:

$$\int_a^b {}_aD_t^{\alpha} f(t)g(t)dt = -\int_a^b f(t)\,{}_tD_b^{\alpha} g(t)dt \qquad (6.8)$$

as long as $f(a)=f(b)=0$ or $g(a)=g(b)=0$.

This formula gives a strong connection between ${}_aD_t^{\alpha}$ and ${}_tD_b^{\alpha}$ via a generalized integration *by part*. This relation is responsible for the emergence of ${}_tD_b^{\alpha}$ in problems of fractional calculus of variations only dealing with ${}_aD_t^{\alpha}$.

As an application to such extended operator, is in finding the solution *f(.)* to minimizes the following fractional problem:

$$V^{\alpha,\beta}_{\mu,(a,b)}(f) = \frac{1}{\Gamma(\alpha)}\int_a^b L(D^{\alpha,\beta}_{\mu} f(s), g(s), s)(t-s)^{\alpha-1} ds$$

by considering a smooth manifold M, L be a smooth Lagrangian function $L{:}C^r \times R^r \times R \rightarrow R$, $r \geq 1$, and any piecewise smooth path $f{:}[a,b] \rightarrow M$, satisfying fixed boundary conditions $f(a)=f_a$ and $f(b)=f_b$. For more details see [1].

References

[1] Agrawal O. P., "*Formulation of Euler-Lagrange equations for fractional variational problems*", J. Math. Anal. Appl., (272), pp. 368-379, (2002).

[2] Diethelm K., Ford N.J., "*Numerical solution of the Bagley-Torvik equation*", BIT 42, pp. 490-507, (2002).

[3] Diethelm K., Luchko Y.; "*Numerical solution of linear multi-term differential equations of fractional order*", J. Comp. Anal. Appl. (2006).

[4] Erwin V.J., Roop J.P., "*Variational Formulation for the stationary fractional advection dispersion equation*", Numer. Math. P.D.E., (22), pp. 558-576, (2006).

[5] Gorenllo R., Mainard F., "*Fractional Calculus: Integral and Differential equations of fractional order*", Springer-Verlag, pp. 223-276, (1997).

[6] Hilfer R., "*Fractal Dynamics, irreversibility and ergodicity breaking*", Chaos, Soliton & Fractal, (1995).

[7] Hull T.E. et. al., "*Comparing numerical methods for ordinary differential equations*", SIAM J. Numer. Anal. (9), (1972).

[8] Kaya D., "*A reliable method for the numerical solution of the Kinetics problems*", Appl. Math. Comp., (2006).

[9] Khusainov T. D., "*Stability Analysis of a Linear Fractional Delay System*", Diff. Equs., vol. 37, no. 8, pp. 1184-1188, (2001).

[10] Kilbas A., Sriastava H. and Trujiilo J., "*Theory and Applications of Fractional Differential Equations*", Amsterdam, Netherlands: Elsevier, (2006).

[11] Lazarevic M. P., "*Finite time stability analysis of* PD$^{\alpha}$ *fractional control of robotic time –delay systems*", Machanics Res. Comm., Vol.33, pp. 269-279, (2006).

[12] Matignon D., "*Stability Result on Fractional Differential Equations with Applications to Control Processing*", In IMACS-SMC proceeding, July France, pp. 963-968, (1996).

[13] Matignon D., "*Stability Properties for Generalized Fractional Differential System*", In: Proceeding of Fractional Differential, (1996).

[14] Miller K. S. and Ross B., "*An introduction to fractional calculus and fractional differential equations*", John Wiley and Sons, New York, (1993).

[15] Momani S.M., "*On the existence of solutions of a system of ordinary differential equations of fractional order*", Far East J. Math. Sci. vol.1, no. 2, pp. 265-270, (1999).

[16] Oldham K. B., Spanier J., "*The fractional Calculus*", Academic Press, New York, (1974).

[17] Pondlubny I., "*Fractional Differential Equations*", Academic Press, New York, (1999).

[18] Pondlubny I. and et. al., "*Mittage-Leffler stability of fractional order non-linear dynamic system*", In Proc. Of the 3rd IFAC Workshop on Fractional Differentiation and its Applications. Ankara, Turkey, (2008).

[19] Samko S. G., Kilbas A.A., Marichev O.I., "*Fractional Integral and Derivatives: Theory and applications*", Gordon and Breach, New York, (1998).

[20] Sabatier J. and et. al.; "*An Overview of the CRONE approach in system analysis, modeling and identification*", observation and control. In Proc. Of the 17th World Congress IFAC, Soul, Korea, (2008).

[21] Sayed A. and et. al., "*Numerical solution for multi-fractional (arbitrary) orders differential equations*", Computational and Applied Mathematics, vol. 23, No. 1, pp. 33-54, (2004).

[22] Shawagfeh N.T., "*Analytical approximation solutions for nonlinear fractional differential equations*", Appl. Math. Comput., vol.131, pp. 517-529, (2002).

[23] Tavazoei M. and Haeri M., "*A necessary condition for double scroll attractor existence in fractional order systems*" Physics Letters A, vol. 367, (2007).

[24] Tavazoei M. and Haeri M., "*A note on the stability of fractional order systems*", Math. And Computers in Simulation, Elsevier, (2008).