On Entire Functions Which Share One Value CM with Their First Derivative

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الخلاصة:

في هذا البحث، نحن نتعامل مع مسألة الوحدانية من الدوال الصحيحة التي لها حصة قيمة واحدة CM مع مشتقتها الاولى. برهن الباحث انه اذا كانت دالة كلية غير ثابتة f ومشتقتها f' لها حصة قيمة واحدة $a(\neq 0)$ CM و $f(z)=a$ عندما $f''(z)=a$ عندئذ $f \equiv f'$. هذه النتيجة لها علاقة بنتيجة Jank-Muse- Volkmann.

Abstract.

In this paper, we deal with the problem of uniqueness of entire functions that share one value CM with their first derivative. The author proves that if a non-constant entire function f and its derivative f' share the value $a(\neq 0)$ CM and $f(z)=a$ when $f''(z)=a$ then $f \equiv f'$. This result is related to a result of Jank-Muse-Volkmann.

Keywords: entire functions, sharing values, Nevanlinna's theory, uniqueness theory.

Classification Categories: Math. Subject classification: 30D35

Introduction

In this paper, we assume that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory (see [1], for example). In particular, $S(r,f)$ denote any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ except possibly for a set E of r of finite linear measure $(E \subseteq \bigcup_{n=1}^{\infty}[r_n, r_n']$

and $mesE \le \sum_{n=1}^{\infty}(r_n' - r_n) < +\infty$, or see [1, P.41]). We say two non-constant entire functions f and g share the value a IM (ignoring multiplicity) if f and g have the same a-points, and also they share the value a CM (counting multiplicity), if f and g have the same a-points with the same multiplicity. Let k be a positive integer, we denote by $N_{k)}(r, \frac{1}{f-a})$ and $N_{(k+1}(r, \frac{1}{f-a})$ the counting function of a-points of f with multiplicity $\le k$ and $>k$, respectively In [2] Jank, Muse and Volkmann proved.

Theorem A

Let f be a non-constant entire function. If f and f' share the value $a(\neq 0)$ IM, and $f''(z) = a$ when $f(z) = a$, then $f \equiv f'$.

It is asked naturally whether the condition " $f''(z) = a$ when $f(z) = a$ " in Theorem A can be replaced by the condition "$f(z) = a$ when $f''(z) = a$". The answer is yes, in fact in this paper we shall prove the following theorem:

Theorem 1: Let f be a non-constant entire function. If f and f' share the value $a(\neq 0)$ CM, and $f(z) = a$ when $f''(z) = a$, then $f \equiv f'$.

Remarks

1. from the hypotheses of Theorem A, it follows readily that f and f' share the value $a(\neq 0)$ CM. In view of this Theorem 1 says, in effect, that Theorem A is also true when reversed the condition. " $f''(z) = a$ when $f(z) = a$ " in Theorem A.

2. The following example shows that the condition "$f(z) = a$ when $f''(z) = a$" in Theorem 1 is necessary:

Example 1. Let $f(z) = e^{2z} + \frac{a}{2}$, where a is a non-zero constant. It is easy to know that f and f' share the value a CM, but the conclusion of Theorem 1 is not valid; that is $f \not\equiv f'$.

3. The method of proof of Theorem 1 can be used to prove Theorem A by considering $\widetilde{F}=\dfrac{f''(f'-f)}{(f'-1)(f-1)}$ instead of $F=\dfrac{f'''(f'-f)}{(f''-1)(f-1)}$ in proof Theorem 1.

The main result

Lemma 1[3]. Let f be a non-constant entire function of finite order. If f and $f^{(k)}$ $(k\geq 1)$ share the value $a(\neq 0)$ CM, then

$f^{(k)}-a=c(f-a)$,

For some nonzero constant c.

Proof of Theorem 1

Suppose $a=1$ (the general case follows by considering $\frac{1}{a}f$ instead of f) and that $f\not\equiv f'$. Set

$$F=\frac{f'''(f'-f)}{(f''-1)(f-1)}, \tag{1}$$

which can be written

$$F=\frac{f'''}{f''-1}\cdot\frac{f'}{f-1}-\left(\frac{f'''}{f''-1}-\frac{f'''}{f''}\right)\frac{f''}{f-1}-\frac{f'''}{f''-1},$$

from the fundamental estimate of logarithmic derivative, we find that

$$\begin{aligned}
m(r,F)\leq{}& m\left(r,\frac{f'''}{f''-1}\cdot\frac{f'}{f-1}\right)+m\left(r,-\frac{f'''}{f''-1}\cdot\frac{f''}{f-1}\right)\\
&+m\left(r,\frac{f'''}{f''}\cdot\frac{f''}{f-1}\right)+m\left(r,-\frac{f'''}{f''-1}\right)+\log 4\\
\leq{}& 3m\left(r,\frac{f'''}{f''-1}\right)+m\left(r,\frac{f'}{f-1}\right)+2m(r,-1)+2m\left(r,\frac{f''}{f-1}\right)+m\left(r,\frac{f'''}{f''}\right)+\log 4\\
={}& 3S(r,f''-1)+S(r,f-1)+0+2S(r,f-1)+S(r,f'')+\log 4\\
={}& 3S(r,f)+3S(r,f)+S(r,f)+\log 4=S(r,f)
\end{aligned} \tag{2}$$

Suppose z''_p is a 1-point of $f''(z)$ of multiplicity $p(\geq 1)$. Then from the hypotheses of Theorem 1, it follows that

$$f(z)=1+(z-z''_p)+\frac{1}{2}(z-z''_p)^2+\frac{a_p}{(p+2)(p+1)}(z-z''_p)^{p+2}+\Lambda\ ,a_p\neq 0\quad,\tag{3}$$

$$f'(z)=1+(z-z''_p)+\frac{a_p}{p+1}(z-z''_p)^{p+1}+\Lambda\quad,\tag{4}$$

$$f''(z)=1+a_p(z-z''_p)^p+\Lambda\quad,\tag{5}$$

$$f'''(z)=pa_p(z-z''_p)^{p-1}+\Lambda\quad,\tag{6}$$

By using (3), (4), (5) and (6) in (1), we obtain $F(z''_p)=O(1)$. Since 1 is a shared value of f and f', we know that the 1-points of f are simple. Similarly, if z_1 is a simple 1-point of $f(z)$, we obtain $F(z_1)=O(1)$. Thus $N(r,F)=0$, combining with (2) we get

$$T(r,F)=S(r,f)\,.\tag{7}$$

Since f and f' share the value 1 CM, there is entire function α such that

$$f'-1=e^{\alpha}(f-1)\,.\tag{8}$$

Differentiating (8) twice we obtain

$$f''=e^{\alpha}[\alpha'(f-1)+f']\,;\tag{9}$$

And

$$f'''=e^{\alpha}[(\alpha''+\alpha'^2)(f-1)+2\alpha'f'+f'']\,.\tag{10}$$

Let z''_p be a 1-point of $f''(z)$ of multiplicity $p(\geq 2)$, then from the hypotheses of Theorem 1, we get that $f(z''_p)=f'(z''_p)=f''(z''_p)=1$ and $f'''(z''_p)=0$. But using (10), we see that $2\alpha'(z''_p)+1=0$. If $2\alpha'+1\equiv 0$, then $\alpha(z)=-\frac{1}{2}z+c$ with c constant. Substituting (8), (9) and (10) into (1) gives

$$f-1=\frac{(G-1)[F-G(G-1)]}{G[(G-1)(G^2-\frac{3}{2}G+\frac{1}{4})-F(G-\frac{1}{2})]}\quad,\tag{11}$$

where $G(z)=e^{-\frac{1}{2}z+c}$. Provided that none, and hence neither, of the numerator and denominator on the right of (11) vanishes identically. We now show that the above possibility cannot in fact arise. If it dose we have,

$$(G-1)(G^2-\frac{3}{2}G+\frac{1}{4})-F(G-\frac{1}{2})\equiv 0\quad,\tag{12}$$

and

$$F - G(G-1) \equiv 0 ; \tag{13}$$

and eliminating F between (12) and (13) leads to $G - \frac{1}{4} \equiv 0$, which is impossible. Now from (11),

$T(r, f) = O(T(r, G))$ for $r \to \infty$, $r \notin E$.

This implies that f has finite order. By Lemma 1 with $k = 1$ we have $G \equiv const$ which is impossible. Therefore $2\alpha' + 1 \not\equiv 0$ and so

$$\overline{N}_{(2}(r, \frac{1}{f'' - 1}) \leq N(r, \frac{1}{2\alpha' + 1}) \leq T(r, \alpha') + O(1) \leq S(r, e^{\alpha}) = S(r, f) . \tag{14}$$

From (8), we find that

$$\alpha' = \frac{f''}{f' - 1} - \frac{f'}{f - 1} . \tag{15}$$

Hence, by (3), (4), and (5) with $p = 1$, we have

$$\alpha'(z_1'') = \frac{1}{2}(a_1 - 1) . \tag{16}$$

Again by using (3), (4), (5) and (6) with $p = 1$ in (1), we obtain

$$F(z_1'') = \frac{1}{2}(a_1 - 1) . \tag{17}$$

From (16) and (17), we see that $F(z_1'') = \alpha'(z_1'')$. If $F \equiv \alpha'$, from (8) and (1) we get

$$f'' - 1 = c(1 - e^{-\alpha}) , \tag{18}$$

Where c is a nonzero constant? Substituting (18) and (8) into (9) gives

$$f - 1 = \frac{-e^{-2\alpha}(e^{\alpha} - 1)(e^{\alpha} - c)}{e^{\alpha} + \alpha'} . \tag{19}$$

It is clear that none, and hence neither of the numerator and denominator on the right of (19) vanishes identically. Since f is an entire function, we know that either $\alpha' \equiv -1$ or $\alpha' \equiv -c$. If $\alpha' \equiv -1$, then (19) becomes

$$f - 1 = -e^{-2\alpha}(e^{\alpha} - c) . \tag{20}$$

Differentiating (20) twice we obtain

$$f'' = e^{-\alpha}(4ce^{-\alpha} - 1) , \tag{21}$$

and eliminating f'' between (18) and (21) leads to $(c+1)e^{2\alpha}-(c-1)e^{\alpha}-4c=0$, from this we conclude that $T(r,e^{\alpha})=S(r,e^{\alpha})$ and so α is a constant which contradicts with $\alpha'\equiv -1$. Similarly, if $\alpha'\equiv -c$, then we have a contradiction. Therefore $F\not\equiv\alpha'$ and so

$$N_{1)}(r,\frac{1}{f''-1})\le N(r,\frac{1}{F-\alpha'})\le T(r,\frac{1}{F-\alpha'})$$
$$\le T(r,F)+T(r,\alpha')+O(1)$$
$$=S(r,f)+S(r,e^{\alpha})=S(r,f),$$

by (7). Combining with (14), we get $\overline{N}(r,\frac{1}{f''-1})=S(r,f)$, from this and the second fundamental theorem for f''

$$T(r,f'')\le\overline{N}(r,\frac{1}{f''})+\overline{N}(r,\frac{1}{f''-1})+S(r,f)\le\overline{N}(r,\frac{1}{f''})+S(r,f),$$

we see that $m(r,\frac{1}{f''})=S(r,f)$. Hence

$$m(r,\frac{1}{f-1})\le m(r,\frac{1}{f''})+S(r,f)=S(r,f). \tag{22}$$

We deduce from (8) and (22) that

$$T(r,e^{\alpha})=S(r,f). \tag{23}$$

If we now eliminate f',f'' and f''' between (1), (8), (9) and (10), we arrive at

$$(f-1)e^{\alpha}[F(e^{\alpha}+\alpha')-e^{3\alpha}-(3\alpha'-1)e^{2\alpha}-(\alpha''+\alpha'^2-3\alpha')e^{\alpha}+\alpha''+\alpha'^2]$$
$$=(e^{2\alpha}+2\alpha'e^{\alpha}-F)(e^{\alpha}-1). \tag{24}$$

In (24), if

$$F(e^{\alpha}+\alpha')-e^{3\alpha}-(3\alpha'-1)e^{2\alpha}-(\alpha''+\alpha'^2-3\alpha')e^{\alpha}+\alpha''+\alpha'^2\equiv 0, \tag{25}$$

Then

$$e^{2\alpha}+2\alpha'e^{\alpha}-F\equiv 0, \tag{26}$$

And eliminating F between (25) and (26) we find that

$T(r,e^{\alpha})=S(r,e^{\alpha})$.

This implies $e^{\alpha}\equiv const$. Hence (8) becomes

$$f'-1=c(f-1), \tag{27}$$

Where c is a nonzero constant? If $N(r,\frac{1}{f''-1})\neq 0$, $c=1$ and so $f\equiv f'$. But this contradiction. Therefore $N(r,\frac{1}{f''-1})=0$, so that

$$f''-1=e^{\beta}, \tag{28}$$

Where β is a non-constant entire function? From (27) and (28), it is easy to know that $\beta = const$, which is a contradiction. Therefore, from (24) it follows that

$T(r,f)=O(T(r,e^{\alpha}))$ For $r\to\infty$, $r\notin E$.

Combining with (23) we see that $T(r,f)=S(r,f)$. This is also a contradiction. The proof of Theorem 1 is complete. ■

References

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