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Approximated Solution of Higher–Order Linear Fredholm Integro–Differential Equations by Computing of Singular Value Decomposition (SVD)

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Abstract

Our main concern here is to give an approximate scheme to solve a linear Fredholm integro–differential equations of higher–order (LFIDE) using expansion method with the expansion functions as basis functions associated with weighted residual technique (collocation method). Computing of singular value decomposition (SVD) has been used to treat these equations approximately. For this method a program is written in matlab (ver.6.5), examples are solved, results are tabulated and comparison is made between the exact and the approximate solution depending on least squares error method.

Keywords: linear Fredholm integro–differential equations, expansion method, and singular value decomposition.

الحل التقريبي لمعادلات خطية تكاملية تفاضلية ذات الرتب العليا من نوع فريدهولم بحساب (SVD)

الخلاصة

هدفنا الرئيسي هو إعطاء مخطط تقريبي لحل معادلات خطية تكاملية تفاضلية ذات الرتب العليا من نوع فريدهولم (LFIDE) باستخدام طريقة التوسيع مع دوال التوسيع كدوال أساس تقرن مع تقنية المتبقي المترجح (طريقة التجميع) مع حساب القيمة المفردة المركبة (SVD) استخدمت لمعالجة هذه المعادلات تقريبيا. لهذه الطريقة تم كتابة برنامج في ماتلاب (ver.6.5) ، وحلت امثلة ، والنتائج ادرجت في جداول وأجريت مقارنة بين الحل الدقيق والحل التقريبي اعتمادا على طريقة اصغر مربعات الخطأ.

1. Introduction

In the recent years many problems of mathematical physics, theory of elasticity, viscodynamics fluid and mixed problems mechanics of continuous media reduce to Fredholm integral and integro–differential equation [1]. The study of numerical methods for integro–differential equations become a topic of considerable interest. Also, the solution form of it is often more suitable for today's extremely fast machine computations [2].

Consider the following form of higher order linear Fredholm integro–differential equation (LFIDE):

$$[D^n + \sum_{i=0}^{n-1} p_i(x) D^i] u(x) = f(x) + \int_a^b k(x,y)\, u(y)\, dy \qquad x, y \in [a,b] \quad ...(1)$$

with two–point boundary conditions:

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$$\sum_{i=0}^{n-1}[\, r_{ji}\, D^{i}\, u(x)\big|_{x=a} + r_{jn+i}\, D^{i}\, u(x)\big|_{x=b}\,] = \alpha_j \qquad j = 0,\ldots, n-1$$

where $k(x,y)$, $f(x)$, $p_i(x)$, $i = 0,\ldots,n-1$ are known functions, $u(x)$ is the unknown function, and $D^i u(x)$ denote the i^{th} derivative of $u(x)$ with respect to x [3] .

In this paper we try to represent an approximate method for solving higher order (LFIDE) in equation (1) using expansion method with the aid of weighted residual technique collocation method [4]. Moreover computing of singular value decomposition (SVD) has been used to determine the values of variables from the resulting normal equations.

The (SVD) is an important tool in numerous applications, it is a well studied, and good numerical method that is available. A singular value decomposition of matrices A is a factorization A = LDU where L and U are orthogonal and D is diagonal as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \lambda_{21} & 1 & 0 & 0 & 0 \\ \lambda_{31} & \lambda_{32} & 1 & 0 & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 1 & 0 \\ \lambda_{51} & \lambda_{52} & \lambda_{53} & \lambda_{54} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{13} \\ 0 & 0 & 1 \end{bmatrix}$$

The diagonal entries of D are allowed by either positive or negative and to appear in any order, but this (SVD) is not unique. We are using a generalized inverse for the computating unique (SVD) [5,6].

2.Expansion Method and Weighted Residual Methods

In this section, we shall discuss and illustrate an important approache coming from the field of approximation theory, this method is called expansion method in which the unknown solution $u(x)$ is expanded in terms of a set of known functions $\varphi_m(x)$, such that [7]:

$$u_N(x) = \sum_{m=0}^{N} c_m \varphi_m(x) \qquad \ldots(2)$$

where c_m are parameters to be determined and $\varphi_m(x)$ are the expansion functions to be chosen.

Now, we present the weighted residual methods by considering the following functional equation:

$$L[\, u(x)\,] = f(x) \qquad x \in D \qquad \ldots(3)$$

where , L denotes an operator which maps $L : U \rightarrow F$ (where U and F are sets of functions) such that $u \in U$, $f \in F$ and D is a prescribed domain.

The approximate solution in equation(2) is substituted in equation (3) for u(x), we obtain the residual defined by :

$$R_N(x) = L[\, u_N(x)\,] - f(x) \qquad \ldots(4)$$

We hope that the residue $R_N(x)$ will become smaller and the exact solution is obtained when the residue is identically zero but this is difficult so we shall try to minimize $R_N(x)$ in some sense.

Now, to minimize the residual $R_N(x)$ the weighted integral of the residue is put equal to zero, that is

$$\int_D w_j\, R_N(x)\, dx = 0 \quad j = 0,1,\ldots,n \quad \ldots(5)$$

where w_j is a prescribed weighting function. Different choices of w_j

yield different methods (Collocation method, Galerkin's method, Partition method , ...) with different approximate solutions.

3. Collocation Method

Here, we discuss a collocation method of the weighted residual method. It is a simple technique for obtaining a linear approximation $u_N(x)$, the weight function w_j in equation (5) are defined as [4]:-

$$w_j = \delta(x-x_j) \qquad \ldots(6)$$

where the fixed points $x_j \in D$, $j = 0, 1, \ldots, n$ are called collocation points and $\delta(x-x_j)$ is Dirac's delta function defined as:

$$\delta(x-x_j) = \begin{cases} 1 & \text{if } x = x_j \\ 0 & \text{elsewhere} \end{cases}$$

Inserting equation(6) in equation(5) gives

$$\int_D \delta(x-x_j) R_N(x_j)\, dx = R_N(x_j) = 0$$

$$j = 0, 1, \ldots, n \qquad \ldots(7)$$

4. Singular Value Decomposition

The singular value decomposition (SVD) of a constant matrix $A \in IR^{p\times q}$, $p \geq q$ is a factorization $A_{p\times q} = L_{p\times p} D_{p\times q} U_{q\times q}$ [8].

where

$$D = \begin{bmatrix} D_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \qquad \ldots(8)$$

and both L (lower triangular) and U (upper triangular) are non singular (even A is singular) [9]. The diagonal matrix $D_{r\times r}$ has dimension and rank r corresponding to the rank A. we can diagonalizable any matrix as.

$$D = L^{-1} A U^{-1} \qquad \ldots(9)$$

Now define a new matrix D^{1} created by taking the inverses of the non–zero (diagonal) elements of D:

$$D^{-1} = \begin{bmatrix} D^{-1}_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} \qquad \ldots(10)$$

D^{-1} is a generalized inverse of D because of the extra structure required. Note that this is a generalized inverse not unique generalized inverse since the matrices on the right side of equation (8) are not–unique. By rearranging equation (8) and using equation (10) we can define a new $(q\times p)$ matrix:

$$G = U^{-1} D^{-1} L^{-1} = A^{-1} \qquad \ldots(11)$$

The importance of the generalized inverse matrix G is revealed in the following theorem.

Theorem (Moore 1920)[10]:

G is a generalized inverse of A since AGA = A.

Moore (1920) and (unaware of Moore's work) Penrose (1955) [11] reduced the infinity of generalized inverse to one unique solution. That is if

1- General condition AGA = A.
2- Reflexive condition GAG = G.
3- Normalized condition $(AG)^T = GA$.
4- Reverse normalized condition $(GA)^T = AG$

Then G matrix is unique.

5.Solution of LFIDE By Expansion Methods Using Weighted Residual Methods

In this section, the solution of (LFIDE) has been found approximately by expansion method using the weighted residual methods.

Recall equation (1)

$$[D^n + \sum_{i=0}^{n-1} p_i(x) D^i] u(x) = f(x) +$$

$$\int_a^b k(x,y)\, u(y)\, dy \qquad x, y \in [a,b]$$

with two–point boundary conditions

$u(a) = u_0,\ u'(x)|_{x=a} = u_1, \dots, u^{(n-1)}(x)|_{x=a} = u_{n-1}$
and $u(b) = v_0,\ u'(x)|_{x=b} = v_1, \dots, u^{(n-1)}(x)|_{x=b} = v_{n-1}$

In operator form, the above equation can be written as.

$$L[u(x)] = f(x)$$

where

$$L[u(x)] = [D^n + \sum_{i=0}^{n-1} p_i(x) D^i] u(x) - \int_a^b k(x,y)\, u(y)\, dy \qquad \dots(12)$$

The unknown function u(x) is approximated by equation(2) where $\varphi_m(x)$ are basic functions to be chosen.

Then from equation(4) and equation (12) we obtain:

$$R_N(x) = \sum_{m=0}^{N} c_m[(D^n + \sum_{i=0}^{n-1} p_i(x) D^i) j_m(x) - \int_a^b k(x,y) j_m(y) dy] - f(x) \dots(13)$$

Define

$$\psi_m(x) = [D^n + \sum_{i=0}^{n-1} p_i(x) D^i] j_m(x) - \int_a^b k(x,y)\, j_m(y)\, dy \qquad \dots(14)$$

Thus (eq.13) becomes:

$$R_N(x, c_0, \dots, c_m) = \sum_{m=0}^{N} c_m \psi_m(x) - f(x) \qquad \dots(15)$$

In this work, we take the basis function $\varphi_m(x) = x^m$.

Now, the problem is how to find the optimal values of c_m's which minimize the residual $R_N(x)$ in equation (15), this can be achieved by using the weighted residual methods with the aid of collocation method.

In this method we choose the collocation points $x_0, \dots, x_N$ in the closed interval [a,b], such that $x_j = a + j\,h$, $j = 0, \dots, N$, where $h = \frac{b-a}{N}$.

Hence, by equation (7) we have

$$\sum_{m=0}^{N} c_m \psi_m(x_j) = f(x_j) \qquad j = 0,1,\dots,N \qquad \dots(16)$$

Equation(16), will provide us with (n+1) simultaneous equations to determine the parameter's c_m's, m = 0, …, N.

Rewrite (equation(16)) in matrix form

$$AC = B \qquad \dots(17)$$

where

$$A = \begin{bmatrix} \psi_{00} & \psi_{01} & \dots & \psi_{0N} \\ \psi_{10} & \psi_{11} & \dots & \psi_{1N} \\ \vdots & & & \\ \psi_{N0} & \psi_{N1} & \dots & \psi_{NN} \end{bmatrix},$$

$$C = \begin{bmatrix} c_0 \\ \vdots \\ c_N \end{bmatrix} \text{ and } B = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_N) \end{bmatrix}$$

In this technique the two-point boundary conditions of equation (1) are added as a new rows in the problem equation (17), these rows can be formed as:

$$u_N^{(k)}(x)\Big|_{x=a} = \sum_{j=0}^{N} c_j j_j^{(k)}(x)\Big|_{x=a} = u_k$$

$$u_N^{(k)}(x)\Big|_{x=b} = \sum_{j=0}^{N} c_j j_j^{(k)}(x)\Big|_{x=b} = v_k$$

$$k = 0, \dots, n-1$$

In matrix form, these equations give:

$\varphi(a)\, C = U$ and $\varphi(b)\, C = V$

Adding these matrices to matrix in equation (17), we obtain :

$$RC = E \qquad \ldots(18)$$

where

$$R = \begin{bmatrix} A \\ \mathbf{L} \\ j\ (a) \\ \mathbf{L} \\ j\ (b) \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} B \\ \mathbf{L} \\ U \\ \mathbf{L} \\ V \end{bmatrix}$$

R and E are constant matrices with the dimensions $p\times q$, $p>q$ and $p\times 1$ respectively, where $q = N+1$ and $p\leq N+2n+1$, depend on the number of the boundary conditions.

Reliable techniques based on singular value decomposition (section 4) can be used to find general inverse of R to determine the solution values c_m which satisfy equation(2), then the approximate solution of equation(1) is given.

6. Examples

The approximate solution of equation (2) can be summarized by the following examples:

Example (1):

Consider the problem, which is 1st order LFIDE:

$$(D+1)\,u(x) = x^2-(2/3)x + 4 + \int_0^2 (x-t)\ u(t)\ dt \qquad ...(19)$$

with boundary conditions $u(0) = 0$, $u(2) = 4$ while the exact is $u(x) = x^2$.

Assume the approximate solution is solution

$$u_2(x)=\sum_{m=0}^{2} c_m x^m = c_0+c_1x+c_2x^2 \qquad ...(20)$$

And hence

$$(D+1)(c_0+c_1x+c_2x^2)=f(x)+ \int_0^2 (x-t)(c_0+c_1t+c_2t^2)\,dt$$

where $f(x) = x^2 - (2/3)x + 4$

So the given equation becomes:

$$(2x+3)\,c_0+(-x+\frac{11}{3})\,c_1+ (x^2-\frac{2}{3}x+4)\,c_2=f(x)$$

That is

$$\psi_0(x)=2x+3,\ \psi_1(x)=-x+\frac{11}{3}$$

$$\text{and} \quad \psi_2(x)=x^2-\frac{2}{3}x+4$$

Then, evaluate $\psi_m(x_j)$, m=0,1,2 and $f(x_j)$, by putting $x_j = a + j\,.h$ and $h = 1$ for all $j=0,1,2$ we have a matrices A and B in equation (17).

Now the boundary conditions are added new rows in the matrices A and B, therefore , we get the matrices R and E in system equation (18) as:

$$R=\begin{bmatrix} 3 & 11/3 & 4 \\ 5 & 8/3 & 13/3 \\ 7 & 5/3 & 20/3 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} \text{ and } E=\begin{bmatrix} 4 \\ 13/3 \\ 20/3 \\ 0 \\ 4 \end{bmatrix}$$

Solve the system equation (18) using singular value decomposition to find general inverse of R (R^+) to determine the coefficients c_0, c_1, c_2 .

At first we write a factorization of R as:

$$R_{5\times3} = L_{5\times5}\, D_{5\times3}\, U_{3\times3}$$

By using equation(11) and equation (10) we obtain :

$$R^{+}=U^{-1}D^{-1}L^{-1}=\begin{bmatrix} -95/186 & 80/93 & -47/186 & 0 & 0 \\ 9/62 & 12/31 & -21/62 & 0 & 0 \\ 1/2 & -1 & 1/2 & 0 & 0 \end{bmatrix}$$

And we get

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = R^{+}E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Substitute c_m's in transforming form $u_2(x)$ to obtain the approximate solution of u(x). Table(1) present a comparison between the exact and approximate solutions depending on least square error method.

Example (2):

Consider the following problem:

$$u''(x) = e^{x} - (e-1)x - 1 + \int_0^1 (x+t)\, u(t)\, dt$$

and $u(0)= \acute{u}(x)|_{x=0} =1$, $u(1) = \acute{u}(x)|_{x=1} = e$ with exact solution $u(x) = e^{x}$.

Assume the approximate solution is in the form:

$$u_3(x) = \sum_{m=0}^{3} c_m x^m$$

Then we have

$$R=\begin{bmatrix} -1/2 & -1/3 & 7/4 & -1/5 \\ -3/2 & -5/6 & 17/12 & 111/20 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad E=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ e \end{bmatrix}$$

Table(2) lists the result obtained by running program to find the approximate solution of the above equation. Included are the least square errors for comparison. $u_3(x) = (1.000) + (1.000)x + (0.5094)x^2 + 0.2904x^3$

Example (3):

Consider the 3^{rd} order LFIDE problem:

$$(D^3 + x)u(x) = x^2 + 2x - 4/3 + \int_0^1 t\, u(t)\, dt$$

with boundary conditions $u(0) = 2$, $\acute{u}(x)|_{x=0} = 1$ and $u(1) = 3$.

The exact solution $u(x) = x + 2$.

Assume the approximate solution is in the form:

$$u_3(x) = \sum_{m=0}^{3} c_m x^m$$

Then we have

$$R=\begin{bmatrix} 1/2 & 1/3 & 1/4 & -29/5 \\ 1/2 & 2/3 & 3/4 & 34/5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad E=\begin{bmatrix} 4/3 \\ 5/3 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

Table(3) lists the result obtained by running program to find the approximate solution of the above equation. Included are the least square errors for comparison. $u_3(x) = (2.0000) + (1.0000)x + (0.0000)x^2 + 0.0000x^3$

7. Conclusions

It has already been proving that expansion method with weight residual technique (collocation method) is a very powerful advice for solving integral equations and integro-differential equations [4].

In this paper we use this method for solving linear Fredholm integro-differential equations of higher-order with two-point boundary conditions. When we applied this method we obtained constant matrices with the dimensions p×q, p>q, then using the (SVD) to give the inverse of these matrices to find the approximate solution.

The computations associated with the examples in this paper we performed using Matlab (ver.6.5).

8. References

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Table (1) Approximate solution of example (1)

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2	L.S.E
Exact	0.000	0.0400	0.1600	0.3600	0.6400	1.0000	1.4400	1.9600	2.5600	3.2400	4.0000	
$u_2(x)$	0.000	0.0400	0.1600	0.3600	0.6400	1.0000	1.4400	1.9600	2.5600	3.2400	4.0000	0.0000

Table (2) Approximate solution of example (2)

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	L.S.E
Exact	1.000	1.1052	1.2214	1.3499	1.4918	1.6487	1.8221	2.0138	2.2255	2.4596	2.7183	
$u_3(x)$	1.000	1.1054	1.2227	1.3537	1.5001	1.6636	1.8461	2.0492	2.2747	2.5243	2.7998	0.0154

Table (3) Approximate solution of example (3)

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	L.S.E
Exact	2.000	2.1000	2.2000	2.3000	2.4000	2.5000	2.6000	2.7000	2.8000	2.9000	3.000	
$u_3(x)$	2.000	2.1000	2.2000	2.3000	2.4000	2.5000	2.6000	2.7000	2.8000	2.9000	3.000	0.0000