Power Series Method For Solving Nonlinear Volterra Integro-Differential Equations of The Second Kind

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Abstract
In this work, we present the power series method for solving special types of the first order nonlinear Volterra integro-differential equations of the second kind. To show the efficiency of this method, we solve some numerical examples.

Keywords: Integro-differential, power series.

1. Introduction
It is known that the integro-differential equations arise in a great many branches of sciences, for example, in potential theory, acoustics, elasticity, fluid mechanics, theory of population, [4], [3].


The power series method is one of the important methods that can be used to solve the initial value problem of the linear Volterra integro-differential equations of the second kind, [2].

In [7], the power series method is used to solve the nonlinear Volterra integral equations of the second kind of the form:

\[ u(x) = f(x) + \lambda \int_0^x k(x,t) \left( u(t) \right)^p \, dt, \quad p \in \mathbb{N} \]

where \( f \) and \( k \) are known functions, \( \lambda \) is a scalar parameter and \( u \) is the unknown function that must be determined.

Here we use the same method to solve the initial value problem that consists of the first order non-linear Volterra integro-differential equations of the second kind of the form:
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\[ u'(x) = f(x) + \lambda \int_{0}^{x} k(x,t) [u(t)]^p \, dt, \]
\[ p \in \mathbb{N} \quad \text{....(1.a)} \]

together with the initial condition:
\[ u(0) = \alpha \quad \text{.... (1.b)} \]

where \( f \) and \( k \) are known functions, \( \alpha \) is a known constant, \( \lambda \) is a scalar parameter and \( u \) is the unknown function that must be determined.

Consider the initial value problem given by equations (1). Assumed the solution of equations (1) takes the form:
\[ u(x) \cong e_0 + e_1 x + e_2 x^2 \quad \text{....(2)} \]

Then by setting \( x = 0 \) into equation (2) one can get:
\[ u(0) \cong e_0. \]

By using the initial condition given by equation (1.b), one can get:
\[ e_0 = \alpha. \]

Then by differentiating equation (2) with respect to \( x \) and setting \( x = 0 \) in the resulting equation on can have:
\[ u'(0) = e_1. \]

On the other hand, from equation (1.a), one can have:
\[ u'(0) = f(0). \]

Therefore
\[ e_1 = f(0). \]

Thus the approximated solution takes the form:
\[ u(x) \cong \alpha + f(0)x + e_2 x^2 \quad \text{.... (3)} \]

where \( e_2 \) is the unknown parameter that must be determined. To do this, we expand \( k(x,y) \) and \( f(x) \) as a power series. That is,
\[ k(x,t) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} k_{ij} x^i t^j \quad \text{.... (4)} \]

and
\[ f(x) = \sum_{i=0}^{\infty} f_i x^i \quad \text{.... (5)} \]

By substituting equations (3)-(5) into equation (1.a) one can get:
\[ f(0) + 2e_2 x = \sum_{i=0}^{\infty} f_i x^i + \int_{0}^{x} \left( \int_{0}^{t} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} k_{ij} x^i t^j \left[ \alpha + f(0)t + e_2 t^2 \right]^p \, dt \right) t^j dt \]

But

\[ \int_{0}^{x} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} k_{ij} x^i t^j \left[ \alpha + f(0)t + e_2 t^2 \right]^p \, dt \]
\[
\left[ \alpha + f(0)t + e_2t^2 \right]^p = \sum_{k=0}^{p} \binom{p}{k} \alpha^k \left[ f(0)t + e_2t^2 \right]^{p-k} = \sum_{k=0}^{p} \binom{p}{k} \alpha^k t^{p-k} \left[ f(0) + e_2t \right]^{p-k} = \sum_{k=0}^{p} \binom{p}{k} \alpha^k t^{p-k} \sum_{l=0}^{p-k} \binom{p-k}{l} [f(0)]^l [e_2t]^{p-k-l}
\]

Therefore equation (6) becomes
\[
f(0) + 2e_2x = \sum_{i=0}^{\infty} f_i x^i + \int 0^x \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{p}{k} \alpha^k t^{p-k} \sum_{l=0}^{p-k} \binom{p-k}{l} [f(0)]^l [e_2t]^{p-k-l} dt = f_0 + f_1x + f_2x^2 + \mathcal{L} + \int 0^x (k_{00} + k_{10}x + k_{01}t + \mathcal{L}) \left[ f(0) \right]^p [e_2t]^{p-1} + \left[ \binom{p}{0} \sum_{l=0}^{p} \binom{p}{l} [f(0)]^l [e_2t]^{p-l} \right] dt
\]
It is clear that \( f(0) \) and \( Q(x^2) \) is a polynomial of degree greater than or equal two. By neglecting \( Q(x^2) \) and solving the equation

\[
e_2 = \frac{f_1 + k_0 \alpha^p}{2},
\]

the unknown parameter \( e_2 \) is determined and therefore the coefficient of \( x^2 \) in equation (3) is obtained.

By repeating the above procedure \( m-1 \) iterations, a power series of the following form derives:

\[
y(x) = \sum_{i=0}^{m} e_i x^i \quad \ldots \quad (7)
\]
Equation (7) is an approximated solution of the initial value problem given by equations (1).

3. Numerical Examples:

In this section we present two examples that are solved by using power series method. These examples shows the efficiency of this method.

Example (1):

Consider the first order nonlinear integro-differential equation of the second kind:

\[ u'(x) = \frac{\frac{1}{2} x^2 - \frac{1}{9} + \int_0^x (x^2 + t) [u(t)]^3 \, dt}{3} \]

Together with the initial condition:

\[ u(0) = 1 \quad \ldots (8.b) \]

Therefore

\[ f(x) = -e^{-x} + \frac{1}{3} \left( x^2 + x + \frac{1}{3} \right) e^{-3x} - \frac{1}{3} x^2 - \frac{1}{9}, \quad p = 3 \]

and

\[ k(x,t) = x^2 + t. \]

We solve this example by using the power series method. To do this, let \( e_0 = u(0) \) and \( e_1 = u'(0) \). Therefore \( e_0 = 1 \) and \( e_1 = f(0) = -1 \). Assume the solution of the above initial value problem takes the form:

\[ u(x) \cong e_0 + e_1 x + e_2 x^2. \]

Hence

\[ u(x) \cong 1 - x + e_2 x^2. \]

But
\[
\begin{align*}
  k_{ij} &= \begin{cases} 
  1 & \text{for } (i, j) = (0,1) \text{ and } \varepsilon \text{.} 
  
  0 & \text{for } (i, j) = (0,2) \varepsilon \text{.} 
  \end{cases}
  
  \text{Thus } k_{00} = 0. \text{ Therefore } 
  
  e_2 = \frac{f_1 + k_{00} \alpha^9}{2} = \frac{1}{2}.
\end{align*}
\]

In this case

\[
Q(x^3) = \frac{12}{35} x^7 + \frac{5}{8} x^6 + \frac{7}{64} x^8 -
\]

\[
\frac{1}{56} x^9 + \frac{3}{8} x^8 - \frac{7}{10} x^7 -
\]

\[
\frac{1}{11} (x_3)^3 x^{11} - \frac{2}{21} (x_3)^2 x^9 + \frac{29}{84} e_3 x^9 -
\]

\[
\frac{1}{6} (x_3)^2 x^{11} - \frac{1}{10} (x_3)^3 x^{12} +
\]

\[
\frac{9}{40} (x_3)^2 x^{10} - \frac{5}{8} e_3 x^8 + \frac{3}{8} (x_3)^2 x^8 -
\]

\[
\frac{3}{32} e_3 x^{10} + \frac{12}{35} e_3 x^7 - \frac{3}{5} e_3 x^7 +
\]

\[
\frac{1}{4} e_3 x^6 + \left[ -\frac{x^2}{3!} + \frac{x^4}{4!} - L \right] -
\]

\[
\frac{1}{3} x^2 \left[ -3 x + \frac{9}{2!} x^2 - \frac{27}{3!} x^3 + L \right] -
\]

\[
\frac{1}{3} \left[ -3 x + \frac{9}{2!} x^2 - \frac{27}{3!} x^3 + L \right] -
\]

\[
\frac{1}{9} \left[ -\frac{27}{3!} x^3 + \frac{81}{4!} x^4 - L \right].
\]

By neglecting \(Q(x^3)\) then equation (9) becomes

\[
\left( 3 e_3 - \frac{1}{3} + 1 - \frac{1}{2} + \frac{1}{3} \right) x^2 = 0
\]

and hence \(e_3 = -\frac{1}{3!}\). Thus

\[
u(x) \equiv 1 - x + \frac{1}{2} x^2 + e_3 x^3
\]

one can get:

\[
\left( 3 e_3 - \frac{1}{3} + 1 \right) x^2 + Q(x^3) = 0
\]

\[
\text{….}(9)
\]

where

\[
Q(x^3) = \frac{12}{35} x^7 + \frac{5}{8} x^6 + \frac{7}{64} x^8 -
\]

\[
\frac{1}{56} x^9 + \frac{3}{8} x^8 - \frac{7}{10} x^7 -
\]

\[
\frac{1}{11} (x_3)^3 x^{11} - \frac{2}{21} (x_3)^2 x^9 + \frac{29}{84} e_3 x^9 -
\]

\[
\frac{1}{6} (x_3)^2 x^{11} - \frac{1}{10} (x_3)^3 x^{12} +
\]

\[
\frac{9}{40} (x_3)^2 x^{10} - \frac{5}{8} e_3 x^8 + \frac{3}{8} (x_3)^2 x^8 -
\]

\[
\frac{3}{32} e_3 x^{10} + \frac{12}{35} e_3 x^7 - \frac{3}{5} e_3 x^7 +
\]

\[
\frac{1}{4} e_3 x^6 + \left[ -\frac{x^2}{3!} + \frac{x^4}{4!} - L \right] -
\]

\[
\frac{1}{3} x^2 \left[ -3 x + \frac{9}{2!} x^2 - \frac{27}{3!} x^3 + L \right] -
\]

\[
\frac{1}{3} \left[ -3 x + \frac{9}{2!} x^2 - \frac{27}{3!} x^3 + L \right] -
\]

\[
\frac{1}{9} \left[ -\frac{27}{3!} x^3 + \frac{81}{4!} x^4 - L \right].
\]

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\[
\left( 4e_4 - \frac{1}{6} + 1 - \frac{3}{2} + \frac{1}{2} \right) x^3 +
\]

\[Q(x) = 0 \quad \ldots \quad (10)\]

Where

\[Q(x) = \frac{59}{765} x^9 + \frac{7}{320} x^{10} + \]

\[
\frac{5}{1188} x^{11} + \frac{13}{64} x^8 + \frac{3}{8} x^4 - \]

\[
\frac{1}{2160} x^{12} - \frac{2}{5} x^7 - \frac{1}{24} (e_4)^2 x^{14} + \]

\[
\frac{7}{12} x^6 - \frac{3}{5} x^5 - \frac{1}{14} (e_4)^3 x^{14} - \]

\[
\frac{26}{63} e_4 x^9 + \frac{9}{35} e_4 x^7 - \frac{1}{13} (e_4)^3 x^{15} - \]

\[
\frac{14}{143} (e_4)^2 x^{13} - \frac{1}{2} e_4 x^6 - \frac{1}{4} e_4 x^8 - \]

\[
\frac{7}{40} (e_4)^2 x^{12} + \frac{31}{720} e_4 x^{12} - \]

\[
\frac{1}{13} e_4 x^{13} - \frac{2}{33} (e_4)^2 x^{11} - \]

\[
\frac{59}{396} e_4 x^{11} - \frac{3}{10} (e_4)^2 x^{10} + \]

\[
\frac{13}{40} e_4 x^{10} + \left[ \frac{x^4}{4!} + \frac{x^5}{5!} \right] - \]

\[
\frac{1}{3} x \left[ \frac{9}{2} x^2 - \frac{27}{3!} x^3 + L \right] - \]

\[
\frac{1}{3} \left[ \frac{27}{3!} x^3 + \frac{81}{4!} x^4 - L \right] - \]

\[
\frac{1}{9} \left[ \frac{81}{4!} x^4 + \frac{243}{5!} x^5 - L \right] - \]

By neglecting \(Q(x^4)\) then equation (10) becomes

\[
\left( 4e_4 - \frac{1}{6} + 1 - \frac{3}{2} + \frac{1}{2} \right) x^3 = 0
\]

and hence \(e_4 = \frac{1}{4!}\). Thus

\[u(x) \equiv 1 - x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \frac{1}{4!} x^4.\]

By continuing in this manner, one can get:

\[u(x) \equiv 1 - x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \]

\[
\frac{1}{4!} x^4 - \frac{1}{5!} x^5 + \ldots = e^{-x}\]

Note that this approximated solution is the exact solution of the initial value problem given by equations (8).

Example (2):

Consider the first order nonlinear integro-differential equation of the second kind:

\[u'(x) = 3x^2 - \frac{1}{8} x^8 \sin x + \int_0^x t \sin x [u(t)]^2 dt \ldots (11.a)\]

\[u(0)=0 \quad \ldots \quad (11.b)\]

Here \(f(x) = 3x^2 - \frac{1}{8} x^8 \sin x\), \(p=2\) and \(k(x, t) = t \sin x\).

We solve this example by using the power series method. To do this, let \(e_0 = u(0)\) and \(e_1 = u'(0)\). Therefore \(e_0 = 0\) and \(e_1 = f(0) = 0\).

Assume the solution of the above initial value problem takes the form:

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\[ u(x) \equiv e_0 + e_1 x + e_2 x^2 \]

Hence

\[ u(x) \equiv e_2 x^2. \]

But

\[ \sin x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!}. \]

Therefore

\[ f(x) = 3x^2 - \frac{1}{8} x^8 \sin x \]

\[ = 3x^2 - \frac{1}{8} x^8 \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!}. \]

and

\[ k(x,t) = t \sin x = t \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!}. \]

Hence \( f_1 = 0 \) and \( k_{00} = 0 \) and this implies that

\[ e_2 = \frac{f_2 + k_{00} \alpha^0}{2} = 0. \]

In this case

\[
Q(x^2) = -\frac{1}{6} (e_2)^2 x^6 \left[ \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!} \right]
- 3x^2 + \frac{1}{8} x^8 \left[ \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!} \right].
\]

Thus

\[ u(x) \equiv 0. \]

By repeating the above argument for the approximated solution:

\[ u(x) \equiv e_3 x^3 \]

one can get:

\[ (3e_3 - 3) x^2 + Q(x^3) = 0 \]

\[ \text{.... (12)} \]

where

\[ Q(x^3) = -\frac{1}{8} (e_3)^2 x^8 \left[ \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!} \right] + \]

\[ \frac{1}{8} x^8 \left[ \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!} \right] \]

By neglecting \( Q(x^3) \) then equation (12) becomes

\[ (3e_3 - 3) x^2 = 0 \]

and hence \( e_3 = 1 \). Thus

\[ u(x) \equiv x^3. \]

By repeating the above argument for the approximated solution:

\[ u(x) \equiv x^3 + e_4 x^4 \]

one can have:

\[ (4e_4) x^3 + Q(x^4) = 0 \quad \text{.... (13)} \]

where

\[
Q(x^4) = \left[ \frac{1}{10} (e_4)^2 x^{10} + \frac{2}{9} e_4 x^9 \right] + \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!} \]

By neglecting \( Q(x^4) \) then equation (13) becomes
(4 \times e_4) x^3 = 0
and hence \( e_4 = 0 \). Thus
\[ u(x) \equiv x^3. \]
By continuing in this manner, one can get:
\[ u(x) \equiv x^3 + 0x^4 + 0x^5 + \ldots = x^3. \]
Note that this approximated solution is the exact solution of the initial value problem given by equations (11).

Remark(1):
The power series method can be also used to solve the initial value problem that consists of the first order nonlinear Volterra integro-differential equation of the second kind:
\[ u'(x) = f(x) + \int_a^x k(x, t) \left[ u(t) \right]^p \, dt \]
\[ \ldots \ldots (14.a) \]

To do this let \( z = t - a \) then equation (14.a) becomes
\[ u'(x) = f(x) + \int_o^{x-a} k(x, z + a) \left[ u(z + a) \right]^p \, dz \]
\[ \ldots \ldots (14.b) \]

Then by setting \( s = x - a \) in the above equation one can have:
\[ y'(s) = f(s + a) + \int_o^{s+a} k(s + a, z + a) \left[ y(z) \right]^p \, dz \]
\[ \ldots \ldots (15.a) \]
where \( y(s) = u(s + a) \). Thus
\[ y(0) = u(a) = \alpha \]
Therefore the initial value problem given by equations (14) reduces to the initial value problem given by equations (15).

4. References