

## ON M- Hollow modules

*Layla H. Al Omairy\**

Received 1, March, 2009

Accepted 10, November, 2009

### Abstract:

Let  $R$  be associative ring with identity and  $M$  is a non- zero unitary left module over  $R$ .  $M$  is called  $M$ - hollow if every maximal submodule of  $M$  is small submodule of  $M$ . In this paper we study the properties of this kind of modules.

**Key words:** Maximal Submodule, Small Submodule, Hallow Module, Projective Module and Lifting Module.

### Introduction:

Let  $R$  be an associative ring with identity and  $M$  be a non- zero unitary left module over  $R$ . A submodule  $N$  of a module  $M$  is called small submodule of  $M$  denoted by  $N \ll M$ , if  $N+L \neq M$  for any proper submodule  $L$  of  $M$  [1].  $M$  is called hollow module if every proper submodule of  $M$  is small submodule [2]. A proper submodule  $N$  of a module  $M$  is called a maximal submodule in  $M$  if whenever  $K$  is a submodule of  $M$  with  $N < K$  then  $K = M$ .

A module  $P$  is called projective  $R$ -module if for every epimorphism  $\beta : B \rightarrow C$  and every homomorphism  $\psi : P \rightarrow C$  there is a homomorphism  $\lambda : P \rightarrow B$  with  $\psi = \beta \lambda$  [1].

Note that if  $P$  is local projective module then every maximal submodule in  $P$  is a small submodule of  $P$  [3].

In this paper we introduce the notation of  $M$ - hollow module that is a module in which every maximal submodule is small submodule. And we discuss some basic properties of this concept

Further more we introduce in section 3 the notation of  $M$ -lifting module and

study the main properties of this modules.

### 1- M - hollow module

In this Section we introduce the concept of  $M$ -hollow modules and study the basic Properties of this type of modules

#### Definition 1.1

A non -zero module  $M$  is called  $M$ -hollow module, if every maximal submodule of  $M$  is small submodule of  $M$ .

It is clear that every hollow module is  $M$ -hollow .

In the following proposition we give some of the basic properties of  $M$ -hollow modules

#### Proposition 1.2

Let  $M$  be a finitely generated module, then  $M$  is  $M$ -hollow iff  $M$  is hollow .

#### Proposition 1.3

Epimorphic image of  $M$ -hollow module is  $M$ -hollow .

**Proof:** Let  $M$  be  $M$ - hollow and let  $f : M \rightarrow M'$  an epimorphism with  $M'$ . Suppose  $N'$  be a maximal Submodule of  $M'$ , Now  $f^{-1}(N')$  is maximal

\*Civil Eng. Dept / College of Eng. / Baghdad Univ. Iraq

Submodule of  $M$  Since other wise  $f^{-1}(N^1) = M$  and hence  $f(f^{-1}(N^1)) = M^1$  then  $N^1 = M^1$  which is contradiction with  $N^1 < M^1$  thus  $f^{-1}(N^1)$  is Proper submodule therefore  $f^{-1}(N^1) \ll M$  and hence  $f(f^{-1}(N^1)) \ll f(M)$  this means  $N^1 \ll M^1$ .

**Corollary 1.4**

Let  $M$  be a module . If  $M$  is  $M$ -hollow module then  $M/N$  is  $M$ -hollow for every proper submodule  $N$  of  $M$  .

**Proof :** Let  $N$  be a proper submodule of  $M$ -hollow module  $M$  . Let  $\pi : M \rightarrow M/N$  be a natural epimorphism then  $M/N$  is  $M$ -hollow module .

**Proposition 1.5**

Let  $K$  be a small submodule of a module  $M$ . If  $M/K$  is  $M$ -hollow module then  $M$  is  $M$ -hollow.

**Proof :** Suppose  $M/K$  is  $M$ -hollow with  $K \ll M$  and Let  $N$  a maximal submodule of  $M$  with  $M=N+L$  where  $L \leq M$  then  $M/K = (N+K)/K$  implies  $M/K = (N+K)/K + (L+K)/K$ ,  $N+K/K$  is proper submodule of  $M/K$  to show  $N+K/K$  is maximal in  $M/K$ . Suppose  $N+K/K < J/K \leq M/K$  thus  $J/K = M/K$  (since  $N$  is maximal in  $M$  which is means  $J=M$ ).Then  $N+K/K$  is small in  $M/K$  and hence  $L+K/K = M/K$  then  $L+K=M$  but  $K \ll M$  then  $L=M$  .

Let  $M$  be a module . If  $M$  is  $M$ -hollow module then  $M/N$  is  $M$ -hollow for every proper Submodule  $N$  of  $M$  .

**Proof :** Let  $N$  be a proper submodule of  $M$ -hollow module  $M$  . Let  $\pi : M \rightarrow M/N$  be a natural epimorphism then  $M/N$  is  $M$ -hollow module .

**Proposition 1.6**

Let  $M$  be a module then  $M$  is  $M$ -hollow and finitely generated module If and only if  $M$  is cyclic and has unique maximal submodule.

**Proof :** Let  $M$  be finitely generated  $M$ -hollow then  $M = Rx_1 + Rx_2 + \dots + Rx_n$ ,  $x_i \in M$ ,  $i=1,2,\dots,n$ .

If  $M \neq Rx_1$  then  $Rx_1$  is proper submodule of  $M$  thus by [1,prop.2.3.11,p.28]  $\exists N$  maximal submodule of  $M$  s.t  $Rx_1 < N$  but  $M$  is hollow so  $N \ll M$  then  $Rx_1 \ll M$  then  $M = Rx_2 + Rx_3 + \dots + Rx_n$

So we delete the Summand one by one until we have  $M = Rx_i$  for Some  $i$  , then  $M$  is cyclic module.

Suppose  $M_1, M_2$  are two distinct maximal submodules then  $M = M_1 + M_2$  but  $M$  is  $M$ -hollow Thus  $M = M_1$  or  $M = M_2$  which is contradiction . The Converse is clear .

**Lemma 1.7**

Let  $M$  be  $M$ -hollow module which has a maximal submodule  $K$  then  $RadM = K$ .

**Proof :** Let  $L$  be a nother maximal submodule in  $M$  ,then  $K+L = M$ , But  $M$  is  $M$ -hollow then  $K=M$  which is contradiction with maximality of  $K$  , therefore  $RadM = K$ .

An  $R$ - module  $M$  is called local module if  $M$  has a unique maximal submodule  $N$  which contains all proper submodule of  $M$  [2] .

**Proposition 1.8**

Let  $M$  be a local module then  $M$  is  $M$ -hollow and cyclic .

**Proof:** Suppose that  $M$  is a local module, then it has a unique maximal  $N$  which contain all other submodule of  $M$ . Let  $w \in M$  with  $w \notin N$  then  $Rw$  submodule of  $M$  . If  $M \neq Rw$  then  $Rw \leq N$  then  $w \in N$  this is a contradiction , then  $Rw = M$  and hence  $M$  is cyclic , Now if  $N+K = M$  for some  $K < M$  then  $K \leq N$  then  $M = N+K \leq N$  then  $M = N$  which is a contradiction , then  $K = M$ , then  $N \ll M$  hence  $M$  is  $M$ -hollow.

**Proposition 1.9**

Let  $M$  be a module,  $M$  is  $M$ -hollow and  $RadM \neq M$  if and only if  $M$  is  $M$ -hollow and cyclic

**Proof :** Let  $M$  be a  $M$ -hollow module with  $\text{Rad } M \neq M$ , then  $M$  has a maximal submodule and by (Lemma 1.7)  $\text{Rad}M$  is the unique maximal of  $M$  and  $M$  is  $M$ -hollow therefor  $\text{Rad}M \ll M$  and  $M/\text{Rad}M$  is a simple module thus cyclic, then  $M/\text{Rad}M = (m + \text{Rad}M)$  for some  $m \in M$  (we claim that  $M = Rm$ ). Let  $w \in M$  then  $w + \text{Rad}M \in M \setminus \text{Rad}M$  hence there is  $r \in R$  such that  $w + \text{Rad}M = r(m + \text{Rad}M) = r m + \text{Rad}M$  i.e  $w - r m \in \text{Rad}M$  thus  $w - r m = y$  for some  $y \in \text{Rad} M$  thus  $w = y + r m \in Rm + \text{Rad} M$ , hence  $M = Rm + \text{Rad}M$ . But  $\text{Rad}M \ll M$  then  $M = Rm$ .

Conversely, since  $M$  is cyclic then  $M$  is finitely generated and thus  $\text{Rad } M \neq M$ .

**Proposition 1.10**

Let  $M$  be a module,  $M$  is  $M$ -hollow if and only if  $\text{Rad}M$  is a small and maximal in  $M$ .

**Proof :** Let  $\text{Rad}M$  be a small and maximal submodule of in  $M$ . To proof  $M$  is  $M$ -hollow, let  $L$  be a maximal submodule of  $M$ , therefore  $M = L + \text{Rad}M$ . But  $\text{Rad}M$  is small thus  $L = M$  which is contradiction, this imply  $\text{Rad } M$  is the unique maximal submodule of  $M$  & small thus  $M$  is  $M$ -hollow module. The converse is clear by (1.6)

**Definition 1.11 [ 3 ]**

A pair  $(p, f)$  is a projection cover of a module  $M$  in case  $P$  is a Projective module  $f: P \rightarrow M$  where  $f$  is an epimorphism and  $\ker f \ll P$ . ( we call  $P$  itself a projective cover of  $M$  )

**Proposition 1.12**

Let  $f: P \rightarrow M$  be a projective cover of  $M$ , if  $M$  is a  $M$ -hollow module then  $P$  is a  $M$ -hollow.

**Proof :** Let  $M$  be a  $M$ -hollow module and since  $f: P \rightarrow M$  is epimorphism then  $P/\ker f$  is isomorphic to  $M$  and hence it

is  $M$ -hollow and  $\ker f \ll P$ , thus  $P$  is a  $M$ -hollow module (by prop. 1.3 & 1.4 ).

We need the following Lemmas.

**Lemma 1.13 [ 4 ]**

If  $P$  is a projective module, then  $P$  is a local module if and only if  $\text{End}(P)$  is a local ring .

**Lemma 1.14 [ 4 ]**

Let  $M$  be a module,  $M$  is a local module if and only if  $\text{Rad}M$  is a small and maximal in  $M$ .

Now we can prove the following proposition .

**Proposition 1.15**

Let  $P$  be a projective module then the following is equivalent:

- (1)  $P$  has a small and maximal submodule.
- (2)  $\text{Rad } P$  is a small and maximal submodule in  $P$ .
- (3)  $P$  is a local module.
- (4)  $\text{End } (P)$  is a local ring.
- (5)  $P$  is  $M$ -hollow
- (6)  $P$  is a projective cover for a simple module.

Proof:

(1)  $\rightarrow$  (2)

Let  $N$  be a maximal and small submodule in  $P$ , then  $\text{Rad } P \leq N$ . Moreover  $N \ll P$  then  $N \leq P$  and hence  $N = \text{Rad } P$ .

(2)  $\rightarrow$  (3)

$P$  is a local module (1.14)

(3)  $\rightarrow$  (4)

Since  $P$  is a local projective module then  $\text{End } (P)$  is a local ring

(4)  $\rightarrow$  (5)

Let  $N$  be a maximal submodule in  $P$ . We must show that  $N \ll P$ .

Now, since  $P$  is a projective module and  $\text{End } (P)$  is a local ring then  $P$  is a local module (1.12) and hence  $P$  is a hollow module. Thus  $N \ll P$ .

(5)  $\rightarrow$  (6)

Since  $P$  is a projective module then  $\text{Rad } P \neq P$ , i.e.,  $P$  has a maximal submodule, say  $N$ . Now,  $P/N$  is a simple module.

Let  $\pi: P \rightarrow P/N$  be the natural epimorphism. We have  $\ker \pi = N$  and

$N \ll P$  by (4) then  $\pi$  is a projective cover for  $P/N$ .

(6)  $\rightarrow$  (1)

Let  $P$  be a projective cover for a simple module, say  $M$ . So there exists an epimorphism  $g: P \rightarrow M$  such that  $\ker g \ll P$ . We only have to show that  $\ker g$  is a maximal submodule in  $P$ . By first isomorphism theorem  $P/\ker g \cong M$  and  $M$  is a simple module then  $P/\ker g$  is also a simple module and this implies that  $\ker g$  is a maximal submodule in  $P$ .

### 2- M-lifting module:

Recall that a module  $M$  is called lifting if for any submodule  $N$  of  $M$ , there exist submodules  $A, B$  of  $M$  such that  $M = A \oplus B$ ,  $A \leq N$  and  $N \cap B \ll B$  [5]

In the following we introduce M-lifting modules and give some properties of this kind of modules.

#### Definition 2.1

An  $R$ -module  $M$  is called M-lifting if for any maximal submodule  $N$  of  $M$ , there exist submodules  $A, B$  such that  $M = A \oplus B$  with  $A \leq N$  and  $N \cap B \ll B$ .

We easily prove the following

#### Remark 2.2

An  $R$ -module  $M$  is M-lifting if and only if for any maximal  $N \leq M$  there exist  $A, B \leq M$  such that  $M = A \oplus B$  with  $A \leq N$  and  $N \cap B \ll M$ .

It is clear that lifting module is M-lifting. The following proposition is give characterization of M-lifting modules.

#### Proposition 2.3

Let  $M$  be an  $R$ -module the following statements are equivalent.

1-  $M$  is M-Lifting

2- Every maximal Submodule  $N$  of  $M$ ,  $N$  can be written as  $N = A \oplus B$  and  $A$  is a direct summand of  $M$  and  $B \ll M$ .

3- For every maximal submodule  $N$  of  $M$  there exists a direct summand  $K$  of  $M$  such that  $K \leq N$  and  $N/K \ll M/K$ .

**Proof:** (1)  $\Rightarrow$  (2) Let  $N$  be maximal submodule of  $M$ . By condition (1) there exist submodules  $K, H$  of  $M$  such that  $M = K \oplus H$  with  $K \leq N$  and  $N \cap H \ll M$ . Since  $N = N \cap M$  So,  $N = N \cap (K \oplus H) = K \oplus (N \cap H)$ . Assume  $A = K, B = N \cap H$  then  $N = A \oplus B$  where  $A$  is direct summand of  $M$  and  $B \ll M$ .

(2)  $\Rightarrow$  (3) Let  $N$  be a maximal submodule of  $M$ , By condition (2),  $N = A \oplus B$  with  $A$  is a direct summand of  $M$  and  $B \ll M$ .

Assume  $K = A$ , so  $K$  is a direct summand of  $M$ .

To prove  $N/K \ll M/K$  Let  $\Pi: M \rightarrow M/K$  be the natural Projection. Since  $B \ll M$  then  $\Pi(B) \ll M/K$  [1]

We claim that  $\Pi(B) = N/K$ . To show that let  $x \in \Pi(B)$ . so  $x = \Pi(b)$  for some  $b \in B$ , Hence  $x = b + k \in N/K$  because  $B \subseteq N$ , thus  $\Pi(B) \subseteq N/K$ .

Now if  $x \in N/K$ , then  $x = a + b + k$ , where  $a \in A, b \in B$ . But  $A = K$ , hence  $x = b + k \in \Pi(B)$ , then  $N/K \subseteq \Pi(B)$ , thus  $N/K = \Pi(B)$  and hence  $N/K \ll M/K$ .

(3)  $\Rightarrow$  (1) Let  $N$  be a maximal submodule of  $M$ , by (3) There exists a direct summand  $K$  of  $M$  such that  $K \subseteq N$  and  $N/K \ll M/K$ . This implies that  $M = K \oplus H$  for some submodule  $H$  of  $M$ . To show  $N \cap H \ll M$ , since  $N = N \cap M$ , then  $N = N \cap (K \oplus H) = K \oplus (N \cap H)$  (moduler Low). But  $M = K \oplus H$  then  $M/K \cong H$ . Let  $g$  be an isomorphism,  $g: M/K \rightarrow H$  which is defined by  $g(m + K) = h$ , if  $m = k + h$  where  $k \in K, h \in H$ . We claim that  $g(N/K) = N \cap H$ , let  $x \in N/K$  then

$x = n+k$  where  $n \in N$ , since  $n \in N \subseteq M = K \oplus H$ ,  $n = k_1 + h_1$  where  $k_1 \in K$ ,  $h_1 \in H$  and so  $g(n+K) = g(k_1+h_1+K) = h_1$  but  $h_1 = n - k_1$  and  $k_1 \in K \subseteq N$  hence  $h_1 \in N \cap H$ , then  $g(N/K) \subseteq N \cap H$ . Now, Let  $d \in N \cap H$ , then  $d \in H$  and  $g(d+K) = g(0+d+K) = d$  then  $d = g(d+K) \in g(N/K)$  then  $N \cap H \subseteq g(N/K)$  thus  $g(N/K) = N \cap H$ , but  $N/K < M/K$  therefore  $g(N/K) \ll H$  i.e  $N \cap H \ll H$  hence  $N \cap H \ll M$ .

It is known that every hollow module is lifting module [6]. To generalize this statement we give the following proposition.

**Proposition 2.4**

Every  $M$ -hollow module is  $M$ -lifting

**Proof**

Let  $N \leq M$  be maximal, if  $N \neq M$ , then  $N \ll M$  and since  $N = \{0\} \oplus N$  thus by definition 3.1, We get the result.

The converse of proposition 2.4 is not true in general as in the following example .

**Example**

Let  $M$  be  $Z$ -module,  $M = Z_2 \oplus Q$ ,  $N = \{0\} \oplus Q$  is a unique maximal submodule of  $M$ , then it clear that  $M$  is  $M$ -lifting but not  $M$ -hollow.

**References:**

1. Kasch F. 1982. Modules and Rings, Academic Press Inc. London. No.(17), P.208.
2. Felury P. 1974. Hollow Modules and local Endomorphism Rings, Pac. J. Math., 53, 379-385.
3. Wisbauer R., 1991, Foundations of Module and Ring theory, Gordon and Brtach Reading. Vol.(3). P. 351.
4. Ali K. M., 2005, Hollow Modules And Semi Hollow Modules, Thesies College of science, University of Baghdad.
5. Keskin D., 2000, On Lifting Modules, Comm. In Algebra, 28(7), 4327-4340.
6. Keskin D. and Tribak R. On lifting Modules and weak Lifting modules, Kyungpook Math. J 45 (3), 445-453. 2005.

## حول المقاسات المجوفة من النوع $M$

ليلى هاشم هلال العميري\*

\*جامعة بغداد / كلية الهندسة / قسم الهندسة المدنية

### الخلاصة:

لتكن  $R$  حلقة تجميعية ذات عنصر محايد وليكن  $M$  مقاسا احاديا غير صفري ايسر معرف على  $R$ . يقال ان المقاس  $M$  مجوف من النوع  $M$  اذا كان كل مقاس جزئي اعظم من  $M$  يكون مقاسا جزئيا صغيرا في  $M$ . في هذا البحث سندرس خواص هذا النوع من المقاسات.