

New theorems in approximation theory

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Abstract:

The aim of this paper is to prove some results for equivalence of moduli of smoothness in approximation theory, we used a "non uniform" modulus of smoothness $\omega_{\phi}^k(f, t) = \sup_{0 < h \leq t} \left\| \Delta_{\beta}^{-k}(f, x) \right\|_p$ and the weighted Ditzian–Totik moduli of smoothness $\omega_{k, \psi}^{\nu}(f, t) = \sup_{0 < h \leq t} \left\| \psi(\cdot)^{\nu} \Delta_{h\psi(\cdot)}^{-k}(f, x) \right\|_p$ in $L_p[-1, 1]$, $0 < p \leq \infty$ by spline functions, several results are obtained. For example, it is shown that, for any $1 \leq \phi \leq r+1$, $1 \leq p \leq \infty$ the inequality $\left| J \right|^{\nu} \omega_{\phi-\nu}^k(S^{(\nu)}, J)_p \approx \omega_{\phi}^k(S, J)_p$, $1 \leq \nu \leq \min[\phi, m+1]$ is satisfied, finally, similar result for Chebyshev partition and weighted Ditzian–Totik moduli of smoothness are also obtained.

Key words: Univariate splines, moduli of smoothness, "non uniform" modulus of smoothness, weighted Ditzian–Totik moduli of smoothness.

Introduction:

Equivalence moduli of smoothness was proved by Hu and Yu [1] and by Hu [2] for uniform and quasi-uniform partition, respectively. In fact, if we set $m = k - 1$, then corollary (1.5) (with an additional restriction $k \leq r$) becomes theorem 1 in [2], also in the case $k = r + 1$ and $m = r - 1$, corollary (1.5) follows from theorem 2 of [2].

A function S is called a spline if:

- 1- the domain is an interval $[a, b]$.
- 2- $S, S', S'', \dots, S^{(r-1)}$ are all continuous functions on $[a, b]$
- 3- there are points (the knots of S) such that $a < Z_0 < Z_1 < \dots < Z_{n-1} < Z_n = b$ and such that S is a polynomial of

degree $\leq r$ on each subinterval $[Z_i, Z_{i+1}]$

Usually piecewise polynomials from $S_r(Z_n)$ are called "splines" if they possess continuous $(r-1)$ st derivatives. We emphasize that if $S \in S_r(Z_n)$ and $m \in \mathbb{N}$ then $S^{(m)} \in \mathcal{D}_{\max\{r-m, 0\}}(Z_n)$ and $S^{(m)} = 0$ a.e if $m \geq r + 1$.

Now a moduli of smoothness are intended for mathematicians working in approximation theory, numerical analysis and real analysis. Measuring the smoothness of function by differentiability is too crude for many purposes in approximation theory. More suitable measurements are provided by moduli of smoothness.

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$L_p(J), 0 < p \leq \infty$, denotes the space of all measurable function f on J Such that $\|f\|_{L_p(J)} < \infty$.

The k th symmetric of f is given by

$$\Delta_{\beta}^{-k}(f, x, [a, b]) = \begin{cases} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x+i\beta) & \text{if } [x, x+k\beta] \subset [a, b] \\ 0 & \text{other wise} \end{cases}$$

then the k th "non uniform" modulus of smoothness of $f \in L_p[-1,1], 0 < p \leq \infty$ is define by

$$\omega_{\phi}^k(f, t) = \sup_{0 < h \leq t} \left\| \Delta_{\beta}^{-k}(f, x) \right\|_p$$

Where

$$\beta = \beta(x, h) = \sqrt{1-x^2}h + h^2, \phi(x) = \sqrt{1-x^2}$$

and

The weighted Ditzian -Totik moduli of smoothness of a function $f \in L_p[-1,1], 0 < p \leq \infty$ is define by

$$\omega_{k, \psi}^{\nu}(f, t) = \sup_{0 < h \leq t} \left\| \psi(\cdot)^{\nu} \Delta_{h\psi(\cdot)}^{-k}(f, x) \right\|_p, [5]$$

Where $\psi(x) = \sqrt{1-x^2}$

A partition $Z_n = \{Z_0, \dots, Z_n\} = -1 < Z_0 < Z_1 < \dots < Z_{n-1} < Z_n = 1$ of the interval $[-1,1]$ denote the scale of partition Z_n by $\theta = \theta(Z_n) = \max \frac{|J_j \pm 1|}{|J_j|}$ where $J_j = [Z_j, Z_{j+1}]$ with $Z_j = -1, j < 0$ and $Z_j = 1, j > n$.

Auxiliary results

The following properties of the moduli of smoothness are well known[6],[7]

(i) for $f \in L_p(J), 0 < p \leq \infty$, we have

$$\omega_{k+1}(f, t, J)_p \leq 2^{\max\{1, 1/p\}} \omega_k(f, t, J)_p, k \in N \tag{2.1}$$

(ii)

$$\omega_k(f, \lambda t, J)_p \leq C(k, p)(\lambda + 1)^{k-1+\max\{1, 1/p\}}$$

$$\omega_k(f, t, J)_p, \lambda > 0 \tag{2.2}$$

(iii) suppose that $f \in L_p[-1,1], 0 < p < \infty$ and $k, \mu \in N$ then

$$\sum_{j=0}^{n-\mu-1} \omega_k(f, \bigcup_{i=j}^{j+\mu} J_i)_p \leq \begin{cases} c(k, \Delta, \mu, p) \omega_k(f, n^{-1})_p^p & \text{if } z_n = u_n^{\Delta} \\ c(k, \mu, p) \omega_k^{\phi}(f, n^{-1})_p^p & \text{if } z_n = t_n \end{cases} \tag{2.3}$$

Lemma .1 (Whitney's inequality). [8]

For any $f \in L_p[a, b], 0 < p \leq \infty$ such that $q_{k-1} \in \Pi_{k-1}$ such that

$$\|f - q_{k-1}\|_{L_p[a, b]} \leq C \omega_k(f, [a, b]). \tag{2.4}$$

Lemma .2

For any polynomial $q_k \in \Pi_k, 0 < p \leq \infty$ and intervals I and J such that $I \subseteq J$ we have

$$\begin{aligned} |J|^{1/p} \|q_r^{(v)}\|_{L_{\infty}(J)} &\approx \|q_r^{(v)}\|_{L_p(J)} \leq \\ C(r, |J|/|I|, p) |J|^{-v} \|q_r\|_{L_p(I)}, & 0 \leq v \leq r \end{aligned} \tag{2.5}$$

The main result

The following theorem is not satisfied for all $f \in L_p, 0 < p < \infty$ but satisfied for spline functions

Theorem .1 (local estimates)

Let $S \in S_r(Z_n), r \in N$ and $J = [Z_{M_1}, Z_{M_2}]$ with $M_2 - M_1 \leq c_0$ for some constant c_0

Then for $1 \leq \phi \leq r+1, 0 \leq p \leq \infty$ we have

$$\omega_{\phi-v}^k(S^{(v)}, J)_p \leq c(r, \theta, c_0, p) |J|^{-v} \omega_{\phi}^k(S, J)_p$$

for all $v = 1, 2, \dots, \phi$.

proof:

Every where in this proof q_{k-1} denote a polynomial

$$\text{of degree } \leq k-1 \ni \|s - q_{k-1}\|_{L_p(J)} \leq C \omega_{\phi}^k(s, J)_p$$

$$(2.4)$$

for any $1 \leq v \leq \phi$. denoting $s_j = s/J_i$

and using [(2.5), [7]] we have $|\Delta_h^{k-v}(s^{(v)}, x, J)| \leq |\bar{\Delta}_p^{k-v}(s^{(v)}, x, J)|$ since

$$\begin{aligned} \omega_{\phi}^k(f, t) &\approx \omega_{\phi}^k(f, t)_{\infty} \\ &= \left| \bar{\Delta}_p^{k-v}(s^{(v)} - q_{k-1}^{(v)}, x, J) \right| \\ &\leq 2^{k-v} \|s^{(v)} - q_{k-1}^{(v)}\|_{L_{\infty}(J)} \end{aligned}$$

$$\leq C(k) \max_{M_1 \leq j \leq M_2-1} \|s^{(v)} - q_{k-1}^{(v)}\|_{L_{\infty}(J_j)}$$

$$\leq C(r, p) \max |J_j|^{-v-\frac{1}{p}} \|s_j - q_{k-1}\|_{L_p(J_j)}$$

$$\leq C(r, \theta, c_0, p) |J_j|^{-v-\frac{1}{p}} \|s - q_{k-1}\|_{L_p(J_j)}$$

$$\leq C(r, \theta, c_0, p) |J_j|^{-v-\frac{1}{p}} \omega_{\phi}^k(s, J)_p.$$

Corollary .2 (equivalence of moduli of smoothness)

Let $S \in S_r(Z_n) \cap C^m[-1,1]$, $r \in N$ and $J = [Z_{M_1}, Z_{M_2}]$ with $M_2 - M_1 \leq c_0$ for some constant c_0

Then for $1 \leq \phi \leq r+1$, $1 \leq p \leq \infty$ we have $|J|^{-v} \omega_{\phi-v}^k(S^{(v)}, J)_p \approx \omega_{\phi}^k(S, J)_p$, $1 \leq v \leq \min[\phi, m+1]$

Equivalence constants above depend only on r, θ, c_0 .

Suppose that $\delta_{\max} = \delta_{\max}(Z_n) = \max_{0 \leq j \leq n-1} |J_j|$ and $\delta_{\min}(Z_n) = \min_{0 \leq j \leq n-1} |J_j|$ we say that Z_n is Δ -quasi uniform if $\Delta = \delta_{\max}/\delta_{\min}$ is bounded by a constant independent of n , and denote such partition by u_n^{Δ} , if $Z_n = u_n^{\Delta}$ then $2/(n\Delta) \leq \delta_{\min} \leq 2/n < \delta_{\max} < 2\Delta/n$ and $\theta(Z_n) \leq \Delta$

Therefore $\delta_{\min} \approx \delta_{\max} \approx n^{-1}$ with Equivalence constants depending only on Δ .

Theorem.3 (quasi uniform partition)

Let u_n^{Δ} , $n \in N$ be a quasi uniform partition of $[-1,1]$ and let $S \in S_r(u_n^{\Delta})$, $r \in N$ then for any $1 \leq \phi \leq r+1$, $0 < p \leq \infty$ we have $\omega_{\phi-v}^k(S^{(v)}, n^{-1})_p \leq c(r, \Delta, p) n^v \omega_{\phi}^k(S, n^{-1})_p$ for all $v = 1, 2, \dots, \phi$.

proof

Since u_n^{Δ} is a Δ -quasi uniform $\delta_{\max} \approx \delta_{\min} \approx |J_i| \approx n^{-1}$ for all $0 \leq j \leq n-1$ with equivalence constants depending only Δ .

Using (2.2) we have

$$\begin{aligned} \omega_{\phi-v}^k(s^{(v)}, n^{-1})_p &= \omega_{\phi-v}^k(s^{(v)}, \frac{\phi}{n\delta_{\min}}, \frac{\delta_{\min}}{\phi}) \\ &\leq C(\phi, \Delta, p) \omega_{\phi-v}^k(s^{(v)}, \frac{\delta_{\min}}{\phi})_p \end{aligned}$$

Now, for any $0 < \beta < \delta_{\min}/\phi$ and $x \in J_j, 0 \leq j \leq n-1$ such that $\Delta_{\beta}^{\phi-v}(s^{(v)}, x) \neq 0$

All point $x - (\phi - v)\beta / 2 + i\beta, 0 \leq i \leq \phi - v$ are either in $J_{j-1} \cup J_j$ or $J_j \cup J_{j+1}$

$$\begin{aligned} \omega_{\phi-v}^k(s^{(v)}, \delta_{\min}/\phi)_p^p &= \sup_{0 \leq \beta \leq \delta_{\min}/\phi} \left\| \Delta_{\beta}^k(\phi-v)(s^{(v)}, \cdot) \right\|_p^p \\ &= \sup_{0 \leq \beta \leq \delta_{\min}/\phi} \sum_{j=0}^{n-1} \left\| \Delta_{\beta}^k(\phi-v)(s^{(v)}, \cdot) \right\|_p^p \\ &\leq 2 \sup_{0 \leq \beta \leq \delta_{\min}/\phi} \sum_{j=0}^{n-1} \left\| \Delta_{\beta}^k(\phi-v)(s^{(v)}, \cdot) \right\|_{L_p(J_{j-1} \cup J_j)}^p \\ &\leq 2 \sum_{j=1}^{n-1} \omega_{\phi-v}^k(s^{(v)}, J_{j-1} \cup J_j)_p^p \\ &\leq C(r, \Delta, p) \sum_{j=1}^{n-1} |J_{j-1} \cup J_j|^{-vp} \omega_{\phi}^k(s, J_{j-1} \cup J_j)_p^p \\ &\leq C(r, \Delta, p) n^{pv} \sum_{j=1}^{n-1} \omega_{\phi}^k(s, J_{j-1} \cup J_j)_p^p \\ &\leq C(r, \Delta, p) n^{pv} \omega_{\phi}^k(s, n^{-1})_p^p. \end{aligned}$$

Where the last inequality follow from (2.3)

Corollary .4

Let $u_n^{\Delta}, n \in N$ be a quasi uniform partition of $[-1,1]$ and let $S \in S_r(u_n^{\Delta}) \cap C^m[-1,1], r \in N, 0 \leq m \leq r-1$ then for any $1 \leq \phi \leq r+1, 1 \leq p \leq \infty$ we have

$$n^{-v} \omega_{\phi-v}^k(S^{(v)}, n^{-1})_p \approx \omega_{\phi}^k(S, n^{-1})_p, 1 \leq v \leq \min[\phi, m+1]$$

Equivalance constants above depend only on $r, \Delta, .$

Let Z_n is a chebyshev partition, if $Z_n = t_n = (t_i)_{i=0}^n$ where $t_i = \cos(\frac{(n-i)\Pi}{n}), 0 \leq i \leq n.$

Theorem.5 (chebyshev knote)

Let t_n be a chebyshev partition of $[-1,1]$ if $s \in S_r(t_n), r \in N$ then, for any $1 \leq \psi \leq r+1, 0 < p < 1$ we have $\omega_{\psi-v}^k(S^{(v)}, n^{-1})_p \leq c(r, p) n^v \omega_{\psi}^k(S, n^{-1})_p$ for all $1 \leq v \leq \psi.$

Proof:

Chebyshev knote: let $J_{j, \phi-v} = [t_{j-3(\psi-v)}, t_{j+4+3(\psi-v)}],$ denote $J_{j, \psi-v},$ and, for each $0 \leq j \leq n-1$

Let $q_j \in \Pi_{\phi-v-1}$ be such that $\|s^{(v)} - q_j\|_{L_p(J_j)} \leq C \omega_{\phi-v}(s^{(v)}, J_j)_p$

[6], and we assume that $q_j = 0$ if $v = \psi.$ Then using theorem 3.1 and the inequality $\psi(x) \leq n|J_j|, x \in J_j,$ we have

$$\begin{aligned} \omega_{k-v, v}^{\psi}(s^{(v)}, n^{-1})_p &= \sup_{0 < h < n^{-1}} \left\| \psi^v \Delta_{h\psi}^{k-v}(s^{(v)}, \cdot) \right\|_p \\ &= \sup_{0 < h < n^{-1}} \sum_{j=0}^{n-1} \left\| \psi^v \Delta_{h\psi}^{k-v}(s^{(v)}, \cdot) \right\|_{L_p(J_j)} \\ &\leq \sum n^{vp} |J_j|^{vp} \\ &\sup_{0 < h < n^{-1}} \left\| \psi^v \Delta_{h\psi}^{k-v}(s^{(v)} - q_j, \cdot) \right\|_{L_p(J_j)} \\ &\leq C(k, p) n^{vp} \sum_{j=0}^{n-1} |J_j|^{vp} \|s^{(v)} - q_j\|_{L_p(J_j)} \\ &\leq C(k, p) n^{vp} \sum_{j=0}^{n-1} |J_j|^{vp} \omega_{k-v, v}(s^{(v)}, J_j)_p \\ &\leq C(r, p) n^{vp} \sum_{j=0}^{n-1} \omega_k(s, J_j)_p \\ &\leq C(r, p) n^{vp} \omega_k^{\psi}(s, n^{-1})_p \end{aligned}$$

where the last inequality follows from (2.3)

Corollary.6

Let $s \in S_r(t_n) \cap C^m[-1,1]$
 $r \in N, 0 \leq m \leq r-1$ then , for any
 $1 \leq \psi \leq r+1, 0 < p < 1$ we have
 $n^{-\psi} \omega_{\psi^{-\nu}}^k(S^{(\nu)}, n^{-1})_p \approx \omega_{\psi}^k(S, n^{-1})_p$ for
 all $1 \leq \nu \leq \min\{\psi, m+1\}$.

Equivalence constants above depend only on r .

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مبرهنات جديدة في نظرية التقريب

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الخلاصة

الهدف من هذا البحث هو برهان نتائج للتكافؤات بين مقاييس النعومة في نظرية التقريب , استخدمنا مقياس

النعومة غير الطبيعي $\omega_{\phi}^k(f, t) = \sup_{0 < h \leq t} \left\| \Delta_{\beta}^{-k}(f, x) \right\|_p$ ومقياس نعومة ديتنزين الثقيل

في الفضاءات $L_p[-1,1], 0 < p \leq \infty$ لدوال السيلاني , نتائج $\omega_{k,\psi}^{\psi}(f, t) = \sup_{0 < h \leq t} \left\| \psi(\cdot)^{-k} \Delta_{h\psi(\cdot)}^{-k}(f, x) \right\|_p$

عديدة حصلنا عليها مثلاً, بينا المتراجعة $\left| J \right|^{\nu} \omega_{\phi^{-\nu}}^k(S^{(\nu)}, J)_p \approx \omega_{\phi}^k(S, J)_p$ عندما $1 \leq \nu \leq \min[\phi, m+1]$, $1 \leq \phi \leq r+1, 1 \leq p \leq \infty$ ايضاً باستخدام مقياس نعومة ديتنزين الثقيل.