ON SOME SEPARATION AXIOMS OF SUPRATOPOLOGICAL SPACES

[1] Taha H. Jassim, [2] Rana B. Yasseen & [3] Nabeel E. Arif

[1, 3] *Dept. of Mathematics, College of Computer Sciences & Mathematics, University of Tikrit, Tikrit, Iraq*
[2] *Dept. of Mathematics, College of Education For Women, University of Tikrit, Tikrit, Iraq*

Abstract

In this paper , we introduce a some new separation axioms **supra-$T_\circ$-space , supra-T_1-space** and **supra-T_2-space** (S^*-$T_\circ$, S^*-T_1, S^*-T_2 for short), and study the supra-hereditary (S^*-hereditary for short) and supratopological property (S^*-topological property for short) on them.

1- Introduction :

Let X,Y be any topological spaces on which separation axioms $(T_\circ,T_1,T_2)$ are assumed unless explicitly stated [3]. A sub class $\tau^* \subseteq p(x)$ is called supratopology on X if $X \in \tau^*$ and τ^* is closed under arbitrary union. (X,τ^*) is called a supratopological space.

The members of τ^* are called supra open sets [1]. Let (X,τ) be a topological space and τ^* be a supratopology on X , we called τ^* asupratopology associated with τ if $\tau \subseteq \tau^*$, let E be a subset of X. The supra closure (resp. supra interior) of E well be denoted by $S^*CL(E)(resp. S^*\operatorname{int}(E))$. Let $(X,\tau_x{}^*) and (Y,\gamma_y{}^*)$ be supratopological spaces . A function $f: X \to Y$ is an S^*-continuous function if the inverse image of each supra open set in Y is a supra open set in X .[1] . we say that f is a supra open function if and only if the image of any supra open set $E \subset X$ is a supra open set $f(E) \subset Y$ and we say that f is a supra homoeomorphism (S^*-home for short) if and only if f is bijective , f is supra open function and f is S^*-continuous [4].

The product of supratopology is

$$\gamma^*_{XXY} = \left\{ \bigcup_{i,j} u_i^* \times v_j^* , u_i^* \in \gamma^*_X , v_j^* \in \gamma^*_Y \right\} \text{ [2] .}$$

For example let $X = \{a,b,c,d\}, Y = \{1,2\}$, $\tau^*_X = \{X,\{a\},\{b\},\{a,b\}\}$, $\gamma^*_Y = \{Y,\{1\},\{2\}\}$ then

$\tau^*_{X\times Y} = \{X\times Y, X\times\{1\}, X\times\{2\}, \{a\}\times Y, \{a\}\times 1, \{a\}\times\{2\}, \{b\}\times Y, \{b\}\times\{1\}, \{b\}\times\{2\}, \{a,b\}\times Y, \{a,b\}\times\{1\}, \{a,b\}\times\{2\}\}$.

$= \{X\times Y, \{(a,1),(b,1),(c,1),(d,1)\}, \{(a,2),(b,2),(c,2),(d,2)\}, \{(a,1),(a,2)\}, \{(a,1)\},$
$\{(a,2)\}, \{(b,1),(b,2)\}, \{(b,1)\}, \{(b,2)\}, \{(a,1),(a,2),(b,1),(b,2)\}, \{(a,1),(b,1)\}, \{(a,2),(b,2)\}\}$

2- $S^* - T_\circ$ - space induced by supratopology .

Definition 2.1.

A supratopological space (X,τ^*) is called supra $T_\circ$-space and denoted by ($S^* - T_\circ$) if for any distinct pair of points x,y of X there exists one supra open set u^* in τ^* such that $x \in u^*$ but $y \notin u^*$ or $x \notin u^*$ but $y \in u^*$.

Example 2.2.

Let $X = \{a,b,c,d\}$ and $\tau^* = \{X,\{b\},\{a,b\},\{b,d\},\{a,b,d\}\}$ it is clearly (X,τ^*) is $S^* - T_\circ$ -space .

Theorem 2.3.

A supratopological space (X,τ^*) is $S^* - T_\circ$ if and only if for each pair of distinct points x,y of X , $S^*cl(\{x\}) \neq S^*cl(\{y\})$.

Proof:

Sufficiency. suppose that for each pair of distinct points $x,y \in X$, $S^*cl(\{x\}) \neq S^*cl(\{y\})$. Let $w \in X$ such that $w \in S^*cl(\{x\})$ but $w \notin S^*cl(\{y\})$.

We claim that $x \notin S^*cl(\{y\})$. for if $x \in S^*cl(\{y\})$ then $S^*cl(\{x\}) \subset S^*cl(\{y\})$. this contradict the fact that $w \notin S^*cl(\{y\})$. Consequently $x \in [S^*cl(\{y\})]^c$ to which y does not belong .

Necessity let (X,τ^*) be an $S^* - T_\circ$ -space and $x,y \in X, x \neq y, \exists$ supra open set u^* $\ni x \in u^*$ or $y \in u^*$ then u^{*c} is an supra closed set which $x \in u^*$ and $y \in u^{*c}$. Since $S^*cl(\{y\})$ is the smallest supra closed set containing y , $S^*cl(\{y\}) \subset u^{*c}$, and therefore $x \notin S^*cl(\{y\})$. Hence $S^*cl(\{x\}) \neq S^*cl(\{y\})$.

Example 2.4.

Let $X = \{a,b,c,d\}, \tau^* = \{X,\{a\},\{b\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$ and $S^*cl(\{a\}) \neq S^*cl(\{b\})$ then (X,τ^*) is $S^* - T_\circ$ -space .

Definition 2.5.

Let (X,τ^*_X) be a supratopological space , E be a subset of X , then classes τ^*_E of all intersections of E with τ^*-supra open subsets of X belong to τ^* is a topology on E it is called relative supratopology (supra-subspace)

Example 2-6:

Let $X = \{a,b,c,d\}, \tau^*_X = \{X,\{b\},\{a,b\},\{b,d\},\{a,b,d\}\}$ and $E = \{a,b,c\}$ then $\tau^*_E = \{E,\{a,b\},\{b\}\}$ and (E,τ^*_E) is called relative supra subtopological space .

Definition 2.7.
Let (X,τ_X^*) be any supra topological space if p is any property in X , then we called p is S^*-hereditary if every relative supra topological space has property p .

Theorem 2.8.
Let (X,τ^*) be any $S^*-T_\circ-$space , then every relative supratopological space is $S^*-T_\circ$.

Proof:
Let (E,τ_E^*) be a relative supratopological space. To show (E,τ_E^*) is $S^*-T_\circ$-space, let $e_1,e_2 \in E$ and $e_1 \neq e_2$,then $e_1,e_2 \in X$. Because (X,τ^*) is $S^*-T_\circ$, there exist a supra open set $u^* \subseteq X$, such that u^* containing one of e_1,e_2 but not both , now if $e_1 \in u^*$, then $e_1 \in E \cap u^* = u^{**}$ .If $e_2 \in u^*$ then $e_2 \in E \cap u^* = u^{**}$ therefore (E,τ_E^*) is $S^*-T_\circ-$space .

Definition 2.9.
Let $f:(X,\tau_X^*) \to (Y,\gamma_Y^*)$ be any S^*-homeomorphic , let p any property in X we say that p is S^*-topological property if p is appear in Y .

Theorem 2.10.
The property of $S^*-T_\circ$ is topological property .

Proof:
Let $(X,\tau_X^*),(Y,\gamma_Y^*)$ be a supratopological spaces and let $f:(X,\tau_X^*) \to (Y,\gamma_Y^*)$ be a S^*- home , suppose $y_1,y_2 \in Y$ such that $y_1 \neq y_2$. Because f is bijective , there exist $x_1,x_2 \in X$ such that $y_1 = f(x_1) and y_2 = f(x_2)$. But (X,τ_X^*) is a $S^*-T_\circ$-space , so there exist a supra open set u^* such that $x_1 \in u^* and x_2 \notin u^*$ or $x_1 \notin u^* and x_2 \in u^*$. Since f a supra open function , so $f(u^*)$ is a supra open set of Y . By bijectivity of f , we get $y_1 = f(x_1) \in f(u^*)$ and $y_2 = f(x_2) \notin f(u^*)$ or $y_1 = f(x_1) \notin f(u^*)$ and $y_2 = f(x_2) \in f(u^*)$. Therefore , (Y,γ_Y^*) is a $S^*-T_\circ-$space .

Theorem 2.11.
A two supratopological spaces $(X,\tau_X^*),(Y,\gamma_Y^*)$ are $S^*-T_\circ-$spaces if and only if $X \times Y$ is a $S^*-T_\circ-$space .

Proof:
Sufficiency . Let X,Y be a $S^*-T_\circ-$spaces , let $(x_1,y_1),(x_2,y_2) \in X \times Y$ and $(x_1,y_1) \neq (x_2,y_2)$. Thus $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume that $x_1 \neq x_2$. Since X is a $S^*-T_\circ-$space , there exists a supra open set u^* such that $x_1 \in u^*, x_2 \notin u^*$ or $x_1 \notin u^*, x_2 \in u^*$. Now the supra open set $u^* \times Y \in X \times Y$, and $(x_1,y_1) \in u^* \times Y$ or $(x_2,y_2) \in u^* \times Y$ then $X \times Y$ is a $S^*-T_\circ-$space.

Necessity . let $X \times Y$ is a $S^*-T_\circ-$space . To show that X is $S^*-T_\circ-$space , take $x_1,x_2 \in X$ such that $x_1 \neq x_2$, $\exists$ two points $(x_1,y),(x_2,y) \in X \times Y$ by definition of product . Since $x_1 \neq x_2$ then $(x_1,y) \neq (x_2,y)$. But $X \times Y$ is $S^*-T_\circ-$space , so $\exists$ a supra open set $u^* \in X \times Y$ such that $(x_1,y) \in u^*$, and it follow that $(x_2,y) \notin u^*$ or $(x_2,y) \in u^*$, and $(x_1,y) \notin u^*$. There exist two supra open sets u_1^*,u_2^* such that $u_1^* X u_2^* = u$. Thus $x_1 \in u_1^* and y \in u_2^*$ or $x_2 \in u_1^* and y \in u_2^*$. Because $(x_1,y) \in u^*$ and $(x_2,y) \notin u^*$, $x_1 \in u_1^* and x_2 \notin u_1^*$ or $x_2 \in u_1^* and x_1 \notin u_1^*$, consequintly X is $S^*-T_\circ-$space . similarly Y is $S^*-T_\circ-$space .

3 - S^*-T_1 - space induced by supratopology .

Definition 3.1.
A supratopological space (X,τ_X^*) is called supra T_1-space and denoted by (S^*-T_1) if for any distinct pair of points x,y of X there exists two supra open sets u^*, v^* in τ_X^* such that $x \in u^*$, $x \notin v^*$ and $y \in v^*$, $y \notin u^*$.

Remark 3.2.
Every S^*-T_1 - spaces is $S^*-T_\circ-$spaces but the converse is not true .

Example 3.3.
Let $X=\{a,b,c,d\}, \tau^* = \{X,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$ it is clearly (X,τ^*) is $S^*-T_\circ-$space but not S^*-T_1 - space .

Theorem 3.4.
Let (X,τ^*) be any S^*-T_1 -space , then every relative supratopological space (E,τ_E^*) is S^*-T_1- .

Proof:
A supratopology space (X,τ^*) be a S^*-T_1 , let $E \subseteq X$ and let $e_1,e_2 \in X$ such that $e_1 \neq e_2$, $\exists$ two supra open sets u^*, v^* such that $e_1 \in u^*$ and $e_1 \in E$ then $e_1 \in E \cap u^* = u^{**}$ and $e_2 \in v^*$ and $e_2 \in E$ then $e_2 \in E \cap v^* = v^{**}$. Hence (E,τ_E^*) is S^*-T_1 .

Theorem 3.5.
The property of S^*-T_1 - space is topological property.

Proof:
Let $(X,\tau_X^*),(Y,\gamma_Y^*)$ be a supratopological spaces , $f:(X,\tau_X^*) \to (Y,\gamma_Y^*)$ is S^*- home , to show (Y,γ_Y^*) is S^*-T_1 .
let $y_1 \neq y_2 \in Y$,since f is onto, $\exists x_1,x_2 \in X$ such that $y_1 = f(x_1), y_2 = f(x_2)$, since f is one-one , $y_1 \neq y_2$ there exists $x_1 \neq x_2$, since (X,τ^*) is S^*-T_1, $\exists$ two

supra open sets u^*, v^* such that $x_1 \in u^*, x_2 \notin u^*$ and $x_1 \notin v^*, x_2 \in v^*$.
Now function supra open then
$f(x_1) \in f(u^*), \forall x_1 \in f(u^*), x_1 \in u^*$ and
$f(x_2) \in f(v^*), \forall x_2 \in f(v^*), x_2 \in v^*$ hence (Y, γ_Y^*) is $S^* - T_1$-space .

Theorem 3.6.
Let $(X, \tau_X^*), (Y, \gamma_Y^*)$ are $S^* - T_1$ -spaces if and only if $X \times Y$ is a $S^* - T_1$ -space .

Proof:
Sufficiency . Let X, Y be a $S^* - T_1$ -spaces , let $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $(x_1, y_1) \neq (x_2, y_2)$. Thus $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume that $x_1 \neq x_2$. Since X is a $S^* - T_1$ -space , there exists two supra open sets u^*, v^* such that $x_1 \in u^*, x_2 \notin u^*$ and r $x_1 \notin v^*, x_2 \in v^*$. Now the supra open sets $u^* \times Y, v^* \times Y \in X \times Y$, and $(x_1, y_1) \in u^* \times X$ but $(x_1, y_1) \notin v^* \times Y$ and $(x_2, y_2) \in v^* \times Y$ but $(x_2, y_2) \notin u^* \times X$ then $X \times Y$ is a $S^* - T_1$ - space.
Necessity . let $X \times Y$ is a $S^* - T_\circ -$ space . To show that X is $S^* - T_1$ -space , take $x_1, x_2 \in X$ such that $x_1 \neq x_2$, $\exists$ two points $(x_1, y), (x_2, y) \in X \times Y$ by definition of product . Since $x_1 \neq x_2$ then $(x_1, y) \neq (x_2, y)$. But $X \times Y$ is $S^* - T_1$ -space , so $\exists$ a two supra open sets $u^*, v^* \in X \times Y$ such that $(x_1, y) \in u^*$, $(x_2, y) \notin u^*$ and $(x_2, y) \in v^*$, $(x_1, y) \notin v^*$. Now there exist two supra open sets u_1^*, u_2^* such that $u_1^* \times u_2^* = u$, and $x_1 \in u_1^*, y \in u_2^*$, since $(x_1, y) \in u^*$, $(x_2, y) \notin u^*$ and $(x_2, y) \in v^*$, $(x_1, y) \notin v^*$ then $x_1 \in u_1^*, x_2 \notin u_1^*$ and $x_2 \in v_1^*, x_1 \notin v_1^*$ then (X, τ_X^*) is $S^* - T_1$-space .Simarly (Y, γ_Y^*) is $S^* - T_1$- space .

4- $S^* - T_2$ - space induced by supratopology .

Definition 4.1
A supratopological space (X, τ_X^*) is called supra T_2-space and denoted by ($S^* - T_2$) if for any distinct pair of points x, y of X there exists disjoint two supra open sets u^*, v^* in τ^* such that $x \in u^*$ and $y \in v^*$.

Example 4.2.
Let $X = \{a, b, c, d\}$,
$\tau^* = \{X, \{a\}, \{b\}\{c\}, \{a, c\}, \{b, a, d\}, \{b, d, a\}, \{d, a\}, \{b, a\}, \{d, a, c\}, \{b, c, a\}, \{d, c\}, \{b, c, d\}\}$
it is clear that (X, τ_X^*) is $S^* - T_2$ -space .

Remark 4.3.
Every $S^* - T_2$ -space is $S^* - T_1$-space but the converse is not true .

Example 4.4.
Let $X = \{a, b, c, d, e\}$,
$\tau^* = \{X, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}$ it is clearly (X, τ^*) is $S^* - T_1$-space but not $S^* - T_2$ - space .

Theorem 4.5.:
Let (X, τ_X^*) be any $S^* - T_2$ -space , then every relative supratopological space (E, τ_E^*) is $S^* - T_2$ -space .

Proof:
Let (X, τ_X^*) be a relative supratopological space and let $e_1, e_2 \in E$ such that $e_1 \neq e_2$. Since $E \subseteq X$, so $e_1, e_2 \in X$. But (X, τ_X^*) is $S^* - T_2$ -space , thus there exist two disjoint supraopen sets u^*, v^* in X , such that $e_1 \in u^*$, $e_2 \in v^*$ and $u^* \cap v^* = \Phi$. Thus $e_1 \in E \cap u^* \in \tau_E^*$, $e_2 \in E \cap v^* \in \tau_E^*$, and $(E \cap u^*) \cap (E \cap v^*) = \Phi$. Hence (E, τ_E^*) is $S^* - T_2$ - space.

Theorem 4.6.
The property of $S^* - T_2$ -space is topological property.

Proof:
Let $(X, \tau_X^*), (Y, \gamma_Y^*)$ be a supratopological spaces and let $f : (X, \tau_X^*) \to (Y, \gamma_Y^*)$ be a $S^* -$ home , suppose $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Because f are one-one and onto , then $f^{-1}(y_1) \neq f^{-1}(y_2)$, But (X, τ_X^*) is a $S^* - T_2$ -space , so there exist two supra open sets u^*, v^* on X such that $f^{-1}(y_1) \in u^*$, $f^{-1}(y_2) \in v^*$ and $u^* \cap v^* = \Phi$.
We get $y_2 \in f(v^*), y_1 \in f(u^*)$. Since supra open function then $f(u^*)$, $f(v^*)$ are supra open sets. Therefore (Y, γ_Y^*) is $S^* - T_2$ -space .

Theorem 4.7.
A two supratopological space $(X, \tau_X^*), (Y, \gamma_Y^*)$ are $S^* - T_2$ -spaces if and only if $X \times Y$ is a $S^* - T_2$ -space .

Proof:
Sufficiency . Let X, Y be a $S^* - T_2$ -spaces , let $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $(x_1, y_1) \neq (x_2, y_2)$. Thus $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume that $x_1 \neq x_2$. Since X is a $S^* - T_2$ -space , there exists two supra open sets u^*, v^* such that $x_1 \in u^*, x_2 \notin u^*$ and $x_1 \notin v^*, x_2 \in v^*$ and $u^* \cap v^* = \Phi$. Now the supra open sets $u^* \times X, v^* \times Y \in X \times Y$, and $(x_1, y_1) \in u^* \times X$ but $(x_1, y_1) \notin v^* \times Y$ and $(x_2, y_2) \in v^* \times Y$ but $(x_2, y_2) \notin u^* \times X$ and $(u^* \times X) \cap (v^* \times Y) = \Phi$ then $X \times Y$ is $S^* - T_2$ -space .
Necessity . let $X \times Y$ is a $S^* - T_2$ - space . To show that X is $S^* - T_2$ -space , take $x_1, x_2 \in X$ such that $x_1 \neq x_2$, $\exists$ two points $(x_1, y), (x_2, y) \in X \times Y$ by

definition of product . Since $x_1 \neq x_2$ then $(x_1, y) \neq (x_2, y)$. But $X \times Y$ is $S^* - T_1$ -space , so $\exists$ a two supra open sets $u^*, v^* \in X \times Y$ such that $(x_1, y) \in u^*$, $(x_2, y) \notin u^*$ and $(x_2, y) \in v^*$, $(x_1, y) \notin v^*$ and $u^* \cap v^* = \Phi$. Now there exist two supra open sets u_1^*, u_2^* such that $u_1^* \times u_2^* = u$, and $x_1 \in u_1^*, y \in u_2^*$, since $(x_1, y) \in u^*$, $(x_2, y) \notin u^*$ and $(x_2, y) \in v^*$, $(x_1, y) \notin v^*$ such that $v_1^* \times v_2^* = v$ then $x_1 \in u_1^*, x_2 \notin u_1^*$ and $x_2 \in v_1^*, x_1 \notin v_1^*$ then (X, τ_X^*) is $S^* - T_2$ -space .Similarly (Y, γ_Y^*) is $S^* - T_2$ - space .

References

1. A.S.Mashhor , A.A.Allam , F.S.Mahmoud and F.H.Khedr ,on supratopological spaces , Indain J.pure Appl. Math 14(1983),502-510 .
2. J.Dugundji , topology , library of congress catalog card number :66-10940 , printed in the United States of America .
3. M.Caldas , D.N.Geogiou and S.Jafari ,characterizations of low separation axioms via α - open sets and α -closure operator , Bol. Soc. Paran ,Math(35)v211/2(2003),1-14 .
4. W.K.Min , H.S.chang , on M-continuity , Kangweon – Kyungki Math. Jour.6 , no 2(1998),323-329 .