Almost Stability of Modified Iteration Method with Errors for a Fixed Point of Uniformaly L- Lipschitzian

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ABSTRACT:
In this paper, we prove strong convergence theorem of modified Mann iteration sequence with errors for uniformaly L- Lipschitzian mapping in arbitrary Banach space. Our results improve and gernalize the recent results Osilike, Xu and Xie and many others.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

Let X be an arbitrary real Banach space and C be a nonempty subset of X. A mapping T:C→C is called:

(i) Strongly pseudocontractive if there exists k ∈ (0,1) such that

\[ \| x - y \| \leq \| x - y + r(I - T - kI)\|x - (I - T - kI)y\| \] ..........(1.1)

where I is the identity mapping on C and for all x, y ∈ C and r>0.

(ii) Lipschitz if there exists a constant L>0 such that

\[ \| T(x) - T(y) \| \leq L \| x - y \| \] ..........(1.2)

for all x, y ∈ C.

(iii) Uniformaly L-Lipschitzian if there exists a constant L>0 such that

The various mapping appearing in the following Definition (1.1) have been studied widely and deeply by many authors; see e.g., [1-4] for more details.

Definition (1.1): Let X be an arbitrary real Banach space and C be a nonempty subset of X. A mapping T:C→C is called:

(i) Strongly pseudocontractive if there exists k ∈ (0,1) such that
\[
\left\| T^n(x) - T^n(y) \right\| \leq L \| x - y \| \quad \text{......(1.3)}
\]

for all \( x, y \in C \) and \( n \geq 1 \).

We consider the iteration [1]

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n + c_n u_n \quad (n \geq 1)
\]

Where \( \{\alpha_n\}, \{c_n\} \) are sequences in \((0, 1)\) and \( \{u_n\} \) is sequence in \( C \) satisfying \( \sum_{n=1}^{\infty} \|u_n\| < \infty \).

This iteration is known Mann iteration sequence with random errors.

We consider the iteration [1]

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n + c_n u_n \quad (n \geq 1)
\]

\[
y_{n+1} = (1 - \beta_n)x_n + \alpha_n Tx_n + e_n v_n
\]

Where \( \{\alpha_n\}, \{\beta_n\}, \{c_n\}, \{e_n\} \) are sequences in \((0, 1)\) and \( \{u_n\}, \{v_n\} \) are sequences in \( C \) satisfying \( \sum_{n=1}^{\infty} \|u_n\| < \infty \), \( \sum_{n=1}^{\infty} \|v_n\| < \infty \).

This iteration is known Ishikawa iteration sequence with random errors.

Let \( x_1 \in C \) and \( x_{n+1} = f(T, x_n) \) define an iteration procedure which yields a sequence of a points \( \{x_n\} \) in \( C \). Suppose that \( F(T) \neq \emptyset \) and \( \{x_n\} \) converges to a fixed point \( q \) of \( T \).

Let \( \{y_n\} \) be an arbitrary sequence in \( C \) and \( \varepsilon_n = \|y_{n+1} - f(T, y_n)\| \). If \( \lim_{n \to \infty} \varepsilon_n = 0 \) implies \( \lim_{n \to \infty} y_n = q \), then the iteration procedure defined by \( x_{n+1} = f(T, x_n) \) is said to be \( T \)-stable or stable with respect to \( T \). Stability results for several iteration procedures for certain contractive definition have been established in recent papers by several authors (see, [5, 6, 7]).

In 1996, Osilike [8], proved that if \( X \) is uniformly smooth Banach space, \( C \) nonempty closed of \( X \) and \( T:C \to C \) is Lipschitz strongly pseudocontractive mapping with fixed point \( q \) in \( C \), then both the Mann and Ishikawa iteration schemes are stable. Then he extended the results to arbitrary real Banach space in [6].
In 2001, Zeqing, Lili and Shin [9], show that if X is an arbitrary real Banach space and \( T: C \to C \) is a Lipschitz strongly pseudocontractive mapping, then under certain conditions the Ishikawa iterative with errors converges strongly to the unique fixed point of \( T \). We also proved that this iteration procedure is stable with respect to \( T \).

In 2004, Xu and Xie [10], proved necessary and sufficient condition for strongly convergence of Mann iteration process with errors to a fixed point of Lipschitz strongly pseudocontractive mapping in real Banach space.

We consider the iteration
\[
x_i \in C,
\]
\[
x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n + u_n \quad (n \geq 1)
\]

Where \( \{\alpha_n\} \) is sequences in \((0,1)\) and \( \{u_n\} \) is sequence in C satisfying \( \sum_{n=1}^{\infty} \|u_n\| < \infty \).

This iteration is known Modified Mann iteration sequence with random errors.

We consider the iteration [1]

Next we recall the definition stability. Let \( x_i \in C \) and \( x_{n+1} = f(T^n, x_n) \) define an iteration procedure which yields a sequence of a points \( \{x_n\} \) in C. Suppose that \( F(T) \neq \emptyset \) and \( \{x_n\} \) converges to a fixed point \( q \) of \( T \).

Let \( \{y_n\} \) be an arbitrary sequence in C and \( \varepsilon_n = \|y_{n+1} - f(T^n, y_n)\| \). If \( \lim_{n \to \infty} \varepsilon_n = 0 \) implies \( \lim_{n \to \infty} y_n = q \), then the iteration procedure defined by \( x_{n+1} = f(T^n, x_n) \) is said to be \( T^n \)-stable or stable with respect of \( T^n \).

It is our purpose in this paper to show that if X is an arbitrary real Banach space and \( T: X \to X \) is uniformly \( L \)-Lipschitzian mapping, then under certain condition the Modified Mann iterative method with errors converges strongly to the unique fixed point of \( T \). We also prove that this iteration procedure is stable with respect to \( T^n \). Our results generalize most of the results that have appeared recently.

For our result we need the following lemma:

**Lemma 1.2.** [1] Let \( \{\alpha_n\} \) be a nonnegative sequence that satisfies the inequality
\[
\alpha_{n+1} \leq (1 - t_n)\alpha_n + b_n + c_n \quad n > 1
\]

where \( t_n \in [0,1] \) for each \( n \in N \), \( \sum_{n=0}^{\infty} t_n = \infty \) and \( b_n = 0(t_n) \), \( \sum_{n=0}^{\infty} c_n < \infty \). Then \( \alpha_n \to 0 \) as \( n \to \infty \).
2. MAIN RESULT

Theorem 2.1. Let $X$ be an arbitrary real Banach space and $T : X \to X$ is uniformly $L$-Lipschitzian mapping and $T$ satisfies the condition

$$\|x - y\| \leq \|x - y + r[(I - T^n - kI)x - (I - T^n - kI)y]\|$$

where $I$ is the identity mapping on $X$ and for all $x, y \in C, n \geq 1$, $r > 0$ and $k \in (0, 1)$.

If $q$ is a fixed point of $T$ and for arbitrary $x_i \in X$, the Modified Mann iterative sequence with errors defined by (1.4) satisfying

$$0 < \alpha < \alpha_n \leq k[2(L^2 + 3L + 3)]$$

where $L > 1$ is Lipschitz constant of $T$. Then

(1) $\{x_n\}$ converges strongly to unique fixed point $q$ of $T$.

(2) $\{y_n\}$ any sequence in $X$. Then $\{y_n\}$ converges strongly to fixed point $q$ of $T$ if and only if $\varepsilon_n$ converges to 0.

Proof(1): using (1.4), we have

$$x_n = x_{n+1} + \alpha_n x_n - \alpha_n T^n x_n - u_n$$

$$= x_{n+1} + \alpha_n x_n - \alpha_n T^n x_n - x_n + 2\alpha_n x_{n+1} - 2\alpha_n x_n + k \alpha_n x_{n+1} - k \alpha_n x_n + \alpha_n T^n x_n - \alpha_n T^n x_{n+1} - u_n$$

$$= (1 + \alpha_n) x_{n+1} + \alpha_n x_n + \alpha_n (I - T^n - kI) x_{n+1} - (2 - k) \alpha_n x_{n+1} + \alpha_n (T^n x_{n+1} - T^n x_n) - u_n$$

$$= (1 + \alpha_n) x_{n+1} + \alpha_n (I - T^n - kI) x_{n+1} - (2 - k) \alpha_n [ (1 - \alpha_n) x_n + \alpha_n T^n x_n + u_n ]$$

$$+ \alpha_n x_n + \alpha_n (T^n x_{n+1} - T^n x_n) - u_n$$

$$= (1 + \alpha_n) x_{n+1} + \alpha_n (I - T^n - kI) x_{n+1} - (2 - k) (\alpha_n - \alpha_n^2) x_n - (2 - k) \alpha_n^2 T^n x_n$$

$$+ (2 - k) \alpha_n u_n + \alpha_n x_n + \alpha_n (T^n x_{n+1} - T^n x_n) - u_n$$

$$= (1 + \alpha_n) x_{n+1} + \alpha_n (I - T^n - kI) x_{n+1} - 2 \alpha_n x_n - 2 \alpha_n^2 x_n + k \alpha_n x_n - k \alpha_n^2 x_n$$

$$- 2 \alpha_n T^n x_n + k \alpha_n T^n x_n + (2 - k) \alpha_n u_n + \alpha_n x_n + \alpha_n T^n x_{n+1} - \alpha_n T^n x_n - u_n$$

$$= (1 + \alpha_n) x_{n+1} + \alpha_n (I - T^n - kI) x_{n+1} - \alpha_n x_n + (2 - k) \alpha_n^2 (x_n - T^n x_n)$$

$$+ k \alpha_n x_n + \alpha_n (T^n x_{n+1} - T^n x_n) + (2 - k) \alpha_n u_n - u_n$$
therefore

\[
x_n = (1 + \alpha_n)x_{n+1} + \alpha_n (I - T^n - kI)x_{n+1} - (1 - k)\alpha_n x_n + (2 - k)\alpha^2_n (x_n - T^n x_n) \\
+ \alpha_n (T^n x_{n+1} - T^n x_n) + (2 - k)\alpha_n u_n - u_n.
\]

Therefore

\[
x_n - q = (1 + \alpha_n)(x_{n+1} - q) + \alpha_n [(I - T^n - kI)x_{n+1} - (I - T^n - kI)q] \\
+ \alpha_n (T^n x_{n+1} - T^n x_n) - (2 - k)\alpha^2_n (T^n x_n - q) - (1 - k)\alpha_n (x_n - q) \\
+ (2 - k)\alpha^2_n (x_n - q) - [1 + (2 - k)\alpha_n]\|u_n\|.
\]

For all \( n \geq 1 \). Furthermore,

\[
\|x_n - q\| \geq (1 + \alpha_n) \|x_{n+1} - q\| + \frac{\alpha_n}{1 + \alpha_n}[(I - T^n - kI)x_{n+1} - (I - T^n - kI)q] \\
+ \alpha_n \|T^n x_{n+1} - T^n x_n\| - (2 - k)\alpha^2_n \|T^n x_n - q\| - (1 - k)\alpha_n \|x_n - q\| \\
+ (2 - k)\alpha^2_n \|x_n - q\| - [1 + (2 - k)\alpha_n]\|u_n\| \tag{2.3}
\]

By virtue of (1.1) and \( T \) is uniformly \( L \)-Lipschitzian, we have

\[
\|x_n - q\| \geq (1 + \alpha_n) \|x_{n+1} - q\| + L\alpha_n \|x_{n+1} - x_n\| - (1 - k)\alpha_n \|x_n - q\| \\
+ (2 - k)\alpha^2_n \|x_n - q\| - 3M \\
\geq (1 + \alpha_n) \|x_{n+1} - q\| - (1 - k)\alpha_n \|x_n - q\| - (L + 1)(L + 2)\alpha^2_n \|x_n - q\| \\
- (3 + L)M \tag{2.4}
\]

where \( M = \sup\{\|u_n\|: n = 1, 2, \ldots\} \)

It follows for (2.4) and the condition (2.2)

\[
\|x_{n+1} - q\| \leq (1 - \alpha_n + \alpha^2_n)\|x_n - q\| + (1 - k)\alpha_n \|x_n - q\| + (L + 1)(L + 2)\alpha^2_n \|x_n - q\| \\
+ (3 + L)M \\
\leq (1 - k\alpha_n)\|x_n - q\| + \alpha_n \left[\alpha_n (L^2 + 3L + 3) - \frac{k}{2}\right]\|x_n - q\| \\
+ (3 + L)M \\
\leq (1 - \frac{k\alpha_n}{2})\|x_n - q\| + (3 + L)M \tag{2.5}
\]
Set \( t_n = \frac{k\alpha_n}{2} \), \( \alpha_n = \|x_n - q\| \) and \( c_n = (3 + L)M \).

Then we have
\[
\alpha_{n+1} \leq (1-t_n)\alpha_n + b_n + c_n
\]

According to the above argument, it is easy seen that
\[
\sum_{n=1}^{\infty} t_n = \infty, \quad b_n = 0(t_n), \quad \sum_{n=1}^{\infty} c_n < \infty
\]

and so, by lemma (1.2), we have \( \lim_{n \to \infty} \|x_n - q\| = 0 \), i.e., \( \{x_n\} \) converges strongly to fixed point \( q \) of \( T \).

If \( p \) also is a fixed point \( T \), putting \( r = 1 \) in (2.1) we obtain
\[
\|q - p\| \leq (1-k)\|q - p\|,
\]

It implies that \( q = p \).

Proof (2): Suppose that \( \lim_{n \to \infty} \varepsilon_n = 0 \), then
\[
\|y_{n+1} - q\| = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T^n y_n - u_n + (1-\alpha_n)y_n + \alpha_n T^n y_n + u_n - q\|
\leq \varepsilon_n + \|(1-\alpha_n)(y_n - q) + \alpha_n (T^n y_n - q) + u_n\|
\leq \varepsilon_n + \|(1-\alpha_n)(y_n - q) + \alpha_n (T^n y_n - q) + u_n\|
\leq (1 - \frac{k\alpha_n}{2})\|y_n - q\| + (3 + L)M + \varepsilon_n
\]

Set \( t_n = \frac{k\alpha_n}{2} \), \( \alpha_n = \|x_n - q\| \) and \( c_n = (3 + L)M + \varepsilon_n \).

Then we have
\[
\alpha_{n+1} \leq (1-t_n)\alpha_n + b_n + c_n
\]

According to the above argument, it is easy seen that
\[
\sum_{n=1}^{\infty} t_n = \infty, \quad b_n = 0(t_n), \quad \sum_{n=1}^{\infty} c_n < \infty
\]

and so, by lemma (1.2), we have \( \lim_{n \to \infty} \|y_n - q\| = 0 \), i.e., \( \{y_n\} \) converges strongly to fixed point \( q \) of \( T \).

Then the iterative process defined by \( x_{n+1} = f(T^n, x_n) \) is \( T^n \)-stable.
On the contrary, let $\{y_n\}$ converges strongly to fixed point $q$ of $T$. Then

$$
\epsilon_n = \|y_{n+1} - (1 - \alpha_n) y_n - \alpha_n T^\ast y_n - u_n\| \\
\leq \|y_{n+1} - q\| + (1 - \alpha_n) \|y_n - q\| + L\alpha_n \|y_n - q\| + M \to 0.
$$

This implies that $\lim_{n \to \infty} \epsilon_n = 0$. \(\Box\)

REFERENCES


