On Generalized Simple Singular P-injective Rings

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Abstract

A ring $R$ is called GSSP-ring, if for any maximal essential right ideal $M$ of $R$ and any $b \in M$ then $bR/bM$ is p-injective. In this paper we give conditions under which GSSP-ring are strongly regular. Finally, some new characteristic properties of GSSP-ring are given.

1. Introduction:

Throughout this paper, $R$ denotes an associative ring with identity, and all modules are unitary right $R$-modules.

A right $R$-modules $M$ is said to be $p$-injective if, for any principle right ideal $p$ of $R$, any right $R$-homomorphism $f : p \to M$, there exists $y$ in $M$ such that $f(b) = yb$ for all $b \in p$. This concept was introduced by Ming [4]. We Recall that: $(1)$ $R$ is called strongly regular if for every $a$ in $R$, there exists an element $b$ in $R$ such that $a = a^2 b$. see [6], $(2)$ A ring $R$ is said to be ERT-ring if for every essential right ideal of $R$ is a two sided ideal. See [5], $(3)$ An ideal $I$ of the ring $R$ is essential if $I$ has a non-zero intersection with every non-zero ideal of $R$, $(4)$ Let $R$ be a ring such that every maximal right ideal is a two sided ideal, then $R$ is called a quasi-duo ring, see [7], $(5)$ A ring $R$ is called reduced if, $R$ contains no non-zero nilpotent element, see [2], $(6)$ For any element $a$ in $R$, $r(a)$ and $I(a)$ denote the right and left annihilator of a respectively, see [2]. $(7)$ $J(R), Z(R)$ will stand respectively for the Jacobson radical, the left singular ideal. See [1]

2. GSSP-rings:

In this section, some of the definitions and basic properties of GSSP-ring are given and we introduce a generalization of such rings. Following [4], a ring is said to be SSPI-rings, if and only if every simple singular $R$-module is P-injective.

Definition 2-1:

A ring $R$ is called a GSSP-ring (generalized simple singular P-injective) if, for any maximal essential right ideal $M$ of $R$, any $b \in M, bR/bM$ is P-injective.

Following [2] a ring $R$ is said to be abelian if each idempotent element of $R$ is central. Next, we give the following lemma which play the key role in several of our proofs.

Lemma 2-2:

Let $R$ be abelian ring, for any maximal right ideal $M$ of $R$, and for any $a \in M$, if $r(a) \subseteq M$, then $M$ is an essential right ideal of $R$.

Proof:

Let $0 \neq a \in M$ and let $r(a) \subseteq M$. Suppose that $M$ is not essential, then $M$ is direct summand, and hence there exists $0 \neq e = e^2$ in $R$ such that $M = r(e)$. Since $a \in M = r(e)$, then $eae = 0$. Since $R$ is abelian, then $ae = 0$, and $e \in r(a) \subseteq M = r(e)$.

Therefore $e = 0$, a contradiction. Thus $M$ is essential. Now, we introduce the following theorem.

Theorem 2-3:

Let $R$ be a ring GSSP-ring, then any right ideal of $R$ is idempotent.

Proof:

Let $I$ be a right ideal of $R$ and let $a \in I$. If $RaR + r(a) \neq R$. Let $M$ be a maximal right ideal containing $RaR + r(a)$. Then by lemma (2-2), $M$ is essential right ideal of $R$. If $aRa = aM$, then $a = ac$ for some $c$ in $M$ and this implies $a(1-c) = 0$. So, $(1-c) \in r(a) \subseteq M$ whence $1 \in M$, a contradiction.

If $aRa \neq aM$, the right $R$-homomorphism $g : R/M \to aR/aM$ defined by $g(b + M) = ab + aM$ for all $b$ in $R$ implies $R/M \cong aR/aM$. Defined $f : aR \to R/M$ as $a$ right $R$-homomorphism by $f(ax) = x + M$, for all $x$ in $R$, then $f$ is a well define right $R$-homomorphism. Indeed, let $x_1, x_2 \in R$ with $ax_1 = ax_2$ implies $(x_1 - x_2) \in r(a) \subseteq M$, thus $x_1 + M = x_2 + M$.

Hence $f(ax_1) = x_1 + M = x_2 + M = f(ax_2)$.

Since $R/M$ is P-injective, then there exists $c$ in $R$ such that $f(ac) = c + M$ and $f(ax = cx + M)$. yields $1 + M = f(a) = da + M$. for $a$, $d$ in $R$, whence $1 \in M$, a contradiction. Thus $RaR + r(a) = R$. In particular $xay + c = 1$, for some $x, y$ in $R$ and $c$ in $r(a)$, so we have $a = axay + ac = axay + 0$. Therefore $a = axay \in I^2$.

This prove $I = I^2$.

Theorem 2-4:

Let $R$ be ERT and GSSP-ring such that the right annihilator of any element in $R$ is essential. Then:

(1) $R$ is reduced.

(2) $J(R) = 0$.

Proof 1:

Let $0 \neq a \in R$ such that $a^2 = 0$ and let $M$ be a maximal right ideal containing $r(a)$. If $aRa = aM$, then $a = ac$ for some $c$ in $M$ this implies $(1-c) \in r(a) \subseteq M$, whence $1 \in M$, a contradiction. Now, since $R/M \cong aR/aM$, then $R/M$ is P-injective.
Defined \( f : aR \to R/M \) by \( f(ar) = r + M \), for every \( r \) in \( R \). Now, we show that \( f \) is a well defined right \( R \)-homomorphism. Indeed if \( ar_1 = ar_2 \) for every \( r_1, r_2 \) in \( R \). Then \( a(r_1 - r_2) = 0 \), therefore \( (r_1 - r_2) \in r(a) \subseteq M \) and hence \( r_1 + M = r_2 + M \). Since \( R/M \) is \( \mathbb{P} \)-injective, then there exists \( y \) in \( R \) such that \( f(ar) = (y + M)ar \), yields \( 1 + M = f(a) = ya + M \). For some \( y \) in \( R \), so \( (1 - ya) \in M \), but \( ya \in r(a) \) is \( a \) right annihilator and hence it is essential. Since \( R \) is ERT, therefore \( r(a) \) is a two sided ideal so \( ya \in M \), thus \( 1 \in M \), a contradiction. Therefore \( a = 0 \), whence \( R \) is reduced.

**Proof 2:**
Let \( a \in J(R) \). If \( aR + r(a) \subseteq R \). Then there exists a maximal right ideal \( M \) containing \( aR + r(a) \). Since \( a \in M \) and \( r(a) \subseteq M \), then by Theorem (2-4)(1) and lemma (2-2), then \( M \) is essential. If \( aR = aM \), then \( a = ab \) for some \( b \) in \( M \) this implies \( (1-b) \in r(a) \subseteq M \), so \( 1 \in M \), a contradiction. If \( aR = aM \), the right \( R \)-homomorphism \( g : R/M \to aR/aM \) defined by \( g(b + M) = ab + aM \), for all \( b \) in \( R \) implies that \( R/M \cong aR/aM \).

Define \( f : aR \to R/M \) as a right \( R \)-homomorphism by \( f(ax) = x + M \) for all \( X \) in \( R \), since \( R \) is reduced (Theorem 2-4)(1) then clearly \( f \) is a well defined \( R \)-homomorphism, so there exists \( y \) in \( R \) such that \( f(ax) = (y + M)x \). Thus \( 1 + M = f(a) = ya + M \). But \( a \in J \subseteq M \), so \( 1 \in M \), a contradiction.

Therefore \( aR + r(a) = R \). In particular \( ar + d = 1 \). for \( d \in r(a) \). This implies \( a = a^2 r \), since \( a \in J \), then there exists an invertible element \( u \) in \( R \) such that \( (1-ar)u = 1 \), so \( u(a - a^2 r)u = a \). yields \( a = 0 \). This proves that \( J(R) = 0 \).

The following theorem gives the condition of being right GSSP-ring are strongly regular.

**Theorem 2-5:** Let \( R \) be an abelian ring and right quasi-duo ring. If \( R \) is GSSP-ring, then \( R \) is strongly regular.

**Proof:**
Assume that \( 0 \neq a \in R \) such that \( a^2 = 0 \). Then there exists the maximal right ideal \( M \) of \( R \) such that \( aR + r(a) \subseteq M \). Observe that \( M \) must be an essential right ideal of \( R \). (lemma 2-2) . By similar method of proof used in Theorem (2-4)(2), we get \( aR + r(a) = R \). In particular \( ay + d = 1 \) for some \( y \) in \( R \), \( d \) in \( r(a) \), thus we have \( a^2 y = a \). Therefore \( R \) is strongly regular ring.

Before closing this section, we present the following result.

**Proposition 2-6:**
If \( R \) is a quasi-duo, GSSP-ring, then \( Z(R) = 0 \).

**Proof:**
If \( Z(R) \neq (0) \), there exists a non-zero element \( a \) in \( Z(R) \) with \( a^2 = 0 \). We want to prove that \( aR + r(a) = R \). If \( aR + r(a) \neq R \). Let \( M \) be a maximal right ideal of \( R \) containing \( aR + r(a) \). Since \( a \in Z(R) \), then \( r(a) \) is essential right ideal and by lemma (2-2) \( M \) is essential maximal right ideal of \( R \). If \( aR = aM \), then \( a = ab \) for some \( b \) in \( M \) and \( (1-b) \in r(a) \subseteq M \), whence \( 1 \in M \), a contradiction. \( M \neq R \). If \( aR \neq aM \), the right \( R \)-homomorphism \( g : R/M \to aR/aM \) defined by \( g(b + M) = ab + aM \) for all \( b \in R \) implies that \( R/M \cong aR/aM \). since \( aR/aM \) is \( \mathbb{P} \)-injective, then \( R/M \) is \( \mathbb{P} \)-injective.

Consider the canonical mapping \( f : aR \to R/M \), then there exists \( a \) in \( R \) such that \( f(a) = 1 + M = ba + M \) implies \( (1 - ba) \in M \), \( ba \in M \) (because \( M \) is two sided ideal), then \( 1 \in M \), a contradiction. Hence \( aR + r(a) = R \).

In particular \( 1 + ar + d \), \( r \) in \( R \), \( d \) in \( r(a) \), so \( a = a^2 r + ad \). Therefore \( Z(R) = 0 \).
References:


المتعمقة P

في الحلقات المنفردة البسيطة العامة من النمط GSSP

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الملخص

يفسر بالحلقة \( R \) بأنها من النمط GSSP ، إذا كان لأي مثالي أعظم أساسي أيمن \( bM \) من حلقة \( R \) ، فإن النمط P . في هذا البحث تم إعطاء شروط أخرى لكي تكون كل حلقة من النمط GSSP من النمط GSSP.