Applying Lie Group Symmetries to Nonlinear Partial Differential Equations

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Abstract:

In this paper an introduction to classical Lie point symmetry method is presented, review for of the role of this symmetries in solving partial differential equations , and then showing some recent results for the application of classical Lie point symmetry method to some nonlinear partial differential equations and determined the solvable Lie group generators of the point symmetries.

Introduction:

Sophus Lie pioneered the modern approach for studying and finding special solutions of systems of nonlinear partial differential equations (PDEs) at the end of the nineteenth century in 1929. There’s still not general theory for finding analytical solutions to a nonlinear PDEs such as in the case of linear ones, the application of group of transformations to study nonlinear PDEs has to be one of the powerful methods that answer this difficult problem this method have been developed in the past few years by Ovysannikov 1982, Ibragimov(1994 -1996) and P.Olver 1986 , and others further remarks on the historical development of this subject and its applications to nonlinear models occurring in different research can be found for example in (Ames 1965),Bluman and COLE 1969), (Clarkson and Kruskal 1989), (Clarkson,1995),(Hydon 2000), (Stephani 1989).

The theory of continues group of transformations created by Lie become one of the most important tools for geometric and algebraic study of general nonlinear PDEs, and solve this PDEs by using their Lie group and continuous symmetry transformations and their invariance, and also the Symmetry analysis plays an important turn in the theory of differential equation. The original symmetry method for reduction of the order of ordinary differential equations (ODE) and reduction of the number of independent variable for both linear and nonlinear (PDE) probably the most useful point transformation of PDEs are those which form acontinous Lie point symmetry group (the classical Lie point symmetry method (CLS)) the method for determining the symmetry group of differential equation is straightforward and described in several books as

In this section we present the basics behind a continuous transformation by only considering one parameter and n independent variable.[2,3,6,10,13,16]

If we have x = (x1 , x2 , ……, xn) in D R = the set of transformations

\[ x^\epsilon = X(x, \epsilon) \] (1)

Define for each x in D and parameter \( \epsilon \) in S R, with the law of composition of parameters \( \phi(\epsilon, \delta) \) in S, forms A one- parameter Lie group of transformations G must be satisfy the following:

1- for each \( \epsilon \) in S the transformations are one to one and onto D.
2- S with the law of composition \( \phi \) forms a group G .
3- for each x in D, \( x^\epsilon = x \) when \( \epsilon = 0 \) corresponds to the identity of G
4- if \( x^\epsilon = X(x, \epsilon) \) and \( x^\delta = X(x, \delta) \) then \( x^{\epsilon+\delta} = X(x, \phi(\epsilon, \delta)) \).
5- \( \epsilon \) is a continues parameter, S is an interval in R, and \( \epsilon = 0 \) corresponds to the identity element.
6- X is infinity differentiable with respect to x in D and an analytical of \( \epsilon \) in S.
7- \( \phi(\epsilon, \delta) \) is an analytical function.

We expand a one- parameter Lie group of transformation around the identity \( \epsilon = 0 \)

\[ x^\epsilon = x + \epsilon X(x, \epsilon) \] \( \implies \) \( \phi(\epsilon, \delta) = \epsilon X(x, \epsilon) \) \( \implies \) \( \phi(\epsilon, \delta) = \epsilon X(x, \epsilon) \) \( \implies \)

\[ \xi(x) = \frac{\partial X(x, \epsilon)}{\partial \epsilon} \] \( \text{at} \) \( \epsilon = 0 \)

the transformation \( x^\epsilon = x + \epsilon \xi(x) \) (3)

is called the infinitesimal transformation of the Lie group of transformations and the components \( \xi(x) \) are called infinitesimals. The operator

\[ X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} \] (4)

is called the *infinitesimal generator* of the one-parameter Lie group of transformations and called Lie group generator.

The commentator, of two infinitesimal generator X1 and X2 is define as

\[ [X1, X2] = X1X2 - X2X1 \]

and the commentator of any two infinitesimal generator of an \( \gamma \)-parameter Lie group of transformations is also a
Lie group generator for one of its one-parameter subgroup.

Where the Lie algebra $\mathfrak{g}$ is a vector space over some field $f$ with additional law of combination of elements in $\mathfrak{g}$ (the commentator) satisfying the properties: let $X_\alpha X_\beta X_\gamma \in \mathfrak{g}$

1) $[X_\alpha X_\beta] = - [X_\beta X_\alpha]$
2) $[X_\alpha [X_\beta X_\gamma]] + [X_\beta [X_\gamma X_\alpha]] + [X_\gamma [X_\alpha X_\beta]] = 0$

With most importantly closure with respect to commutation.

For example if we have the group of Rigid motion in $\mathbb{R}[15]$

$x^* = x \cos \epsilon - y \sin \epsilon + \epsilon_3$
$y^* = x \sin \epsilon + y \cos \epsilon + \epsilon_3$

the corresponding infinitesimal generator are

$X_1 = \partial_x + \partial_y$
$X_2 = \partial_x$
$X_3 = \partial_y$

The commutator table of its Lie algebra follows:

$$
\begin{array}{ccc}
X_1 & X_2 & X_3 \\
X_1 & 0 & -X_3 & X_2 \\
X_2 & X_3 & 0 & 0 \\
X_3 & -X_2 & 0 & 0 \\
\end{array}
$$

### Lie Group of symmetry Transformations

The present section will present comprehensive method for differential equation via the use of symmetry groups. Symmetry group of differential equations transforms solutions of the system to other solutions before attempting to determine the symmetry groups of systems of differential equations.

Consider the system of lines: $x = cy + d$ and the one-parameter Lie group of transformation:

$G_c: (x,y) \rightarrow (x+cd, y+d); d \in \mathbb{R}, c \text{ is constant.}$ the lines are clearly $G_c$ invariant, and so $G_c$ is asymmetry group of system of lines.

The above example looks at symmetries of the solution of system of equations. we can also examine the invariance of function, we said the surface $F(x,y,u)$.

For example if we have the group of $R_2[15]$:

$x^* = x+\epsilon_1 y$  
$y^* = x+\epsilon_2 y + \epsilon_3$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$ are small parameters. the function $f(x,y)$.

For the translation group $G_c$ the infinitesimal generator

$X = \eta_i \partial_i$  
$\eta_i = \eta_i(x,y)$

are called the symmetry of the solution.

The condition of invariance (the symmetry condition)

$\eta_i + \eta_j \partial_j = \epsilon \eta_i$ 

The above examples shows that symmetry groups of systems of linear and nonlinear PDEs is the classical Lie point symmetry method.

The final topic to be addressed before studying the symmetries of differential equations is the process of prolongation. The prolongation is vector function from the space of the independent variable to the space $U^0$, whose entries represent the values of $f$ and all its derivatives up to order $n$. consider a system with two independent and one dependent variables then the space contains all partial derivatives of $u$.

As an example, the Laplace’s equation in the plane $\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0$ the equation with coordinates $(x,y,u,ux,uy,uxx,uxy,uyy)$. 

Now let we consider the one-parameter Lie group of transformation:

$x^* = x + \epsilon_1 x + \epsilon_2 y + \epsilon_3$  
$y^* = y + \epsilon_4 x + \epsilon_5 y + \epsilon_6$  
$u^* = u + \epsilon_7 x + \epsilon_8 y + \epsilon_9$

For $i=1,\ldots,n$ acting on $(x,y,u)$ – space

An operator

$\eta = \sum_{i=1}^{n} \alpha_i \partial_i$

$X = \sum_{i=1}^{n} \alpha_i \partial_i$  
$\epsilon \eta = \eta_1(x,y,u) + \cdots + \eta_n(x,y,u) + O(\epsilon^2)$

The extended $\eta_i$, $\eta_{ij}$ (i,j=1,2) are in [4,13] then

$X^{[2]}$ has the form:

$$
X^* = X + \eta_1(x,y,u) + \eta_2(x,y,u) + O(\epsilon^2)
$$

the condition of invariance $\eta_1 + \eta_2$ and $\eta_{ij} = \epsilon$ (i,j=1,2)

$\eta_1(x,y,u) + \eta_2(x,y,u) + O(\epsilon^2)$

yields linear system of PDEs in the functions $\{x_1, x_2, \ldots, \eta \}$ which is called the system of determining equations which is solved to obtained an infinitesimals Lie group of symmetries.

To find the similarity reduction and particular solutions to the PDE we solve the charististic equation

$\xi_1 \xi_2 \ldots \eta_0 = 0$

$$
\eta = \sum_{i=1}^{n} \alpha_i \partial_i$

$X = \sum_{i=1}^{n} \alpha_i \partial_i$  
$\epsilon \eta = \eta_1(x,y,u) + \cdots + \eta_n(x,y,u) + O(\epsilon^2)$

the condition of invariance (the symmetry condition)

$\eta_1 + \eta_2 + \cdots + \eta_n = \eta(x_1, x_2, \ldots, x_n, u)$

$\eta_{ij} = \epsilon$ (i,j=1,2)

yields linear system of PDEs in the functions $\{x_1, x_2, \ldots, \eta \}$
and by several integrals we obtain the similarity solution which reduce the PDE to which is called the principle ODE.

**Examples:**

Classical Lie Group Symmetry of Kadomtsev–Petviashvili(KP) equation.

The eq

\[ u_{tt} + 3 u_x y + 6 u u_x x + 6 u_x^2 + u_{xxxx} = 0 \quad (13) \]

Which is one the equations frequently examined in connection with solution procedures for non-linear models. This equation arises in several physical applications ranging from surface waves of rectangular canals to applications in plasma physics.

The point symmetries of the (KP)eq. Were examined in [18], in expanded form Kp eq. In (10) has the operator and we can written the prolongation (extended) of the operator \( X \) of fourth prolongation, the symmetry condition of eq (13) yields a linear system of PDEs in the functions \( \{ \xi_1, \xi_2, \xi_3, \eta \} \) which is called the system of determining equations whose solution is obtained in terms of arbitrary functions \( f(t), f_2(t), f_3(t) \) as:

\[ \xi_1 = f_3(t) + \left( 6x f_1(t) - y(3f(t) + y f l^{-1}(t)) \right) \]

\[ \xi_2 = f_2(t) + \frac{y f l^{-1}(t)}{3} \]

\[ \xi_3 = f l^{-1}(t) \]

\[ \eta = \left( -24 u f_1(t) + 6 f_2(t) + 6 x f l^{-1}(t) - 3 y f l^{-1}(t) - y f l^{-1}(t) \right) \]

Hence we obtained an infinitesimal Lie group of symmetries.

To find similarity reduction and particular solutions to the KP eq we drive subgroup, let

\[ f l(t) = k_5 t + k_6 t^2 + k_7, \]

\[ f_2(t) = k_3 t + k_4 \]

\[ f_3(t) = k_1 t + k_2 \]

yields a seven Lie algebra with basis

\[ X = [2T,0,0], X = [1,0,0], X = [0,1,0], X = 3ty-x, X = 9u t + 12tu + 3x u, X = x, y, z, t ] \]

and the condition of invariance is

\[ X = \xi f(x, t) + \tau \eta f(x, t) + \eta u f l^{-1}(t) \]

The complex modified Korteweg–de Vries–II (CMKdV–II) Equation

The inhomogeneous nonlinear heat equation in the form [12]

\[ \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f(x,u) \]

1- when \( f(x) = x^m \) and \( g(x) = x^n \), then eq.(17) become

\[ \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = \eta \xi u^m \]

Classical Lie Group Symmetry of Nonlinear Model of the Heat equation

The equations was in (21) has the solution in eqs.(20) at \( p = m = 0 \) i.e.,

\[ \xi f(x, t) \]

The nonvanishing commutators are \[ [X_1, X_4] = X_1 \]

\[ [X_2, X_4] = 3X_2 \]

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and the condition of invariance is

\[ \xi = \xi f(x, t) + \tau \eta f(x, t) + \eta u f l^{-1}(t) \]

the system in eq.(16) admits Lie group with generators

\[ X_1 = ax, X_2 = at, X_3 = v \partial_u \sqrt{v}, X_4 = x ax + 3 at - u ax - v \sqrt{v} \]

The nonvanishing commutators are \[ [X_1, X_4] = X_1 \]

\[ [X_2, X_4] = 3X_2 \]

Hence a commutator table can be formed as follows:

\[ X_1, X_2, X_3, X_4 \]

\[ X_1 = 0, 0, 0, X_1 \]

\[ X_2 = 0, 0, 3X_2 \]

\[ X_3 = 0, 0, 0, 0 \]

\[ X_4 = -X_1, -3X_2, 0 \]

**Classical Lie Group Symmetry of Nonlinear Model of the Heat equation**
N = u t + u x x + u x x x = 0 \tag{22}

Lie Point Symmetry of the eq.(22) with the exception of the case where n=1, has been classified in [12] where the symmetry generator is

\[ X = \xi(x,t,u) \frac{\partial}{\partial x} + \tau(x,t,u) \frac{\partial}{\partial t} + \eta(x,t,u) \frac{\partial}{\partial u} \]

The symmetry of eq.(22) summarized in the following table:

<table>
<thead>
<tr>
<th>n</th>
<th>\mu</th>
<th>\left( \xi, \tau, \eta \right)</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>\xi \in \mathbb{R}</td>
<td>(0,1,0)</td>
</tr>
<tr>
<td>X2</td>
<td>arbitrary</td>
<td>(a,2,0)</td>
</tr>
<tr>
<td>X3</td>
<td>\nu(n+2)</td>
<td>\left( (n+2)x,0,-2x \right)</td>
</tr>
<tr>
<td>X4</td>
<td>\xi = -2,-1,0</td>
<td>\frac{-n(2n+2)}{n+2}</td>
</tr>
<tr>
<td>X5</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

The Lie Group Symmetry Algebra for Some Nonlinear PDEs

1- the Boussinesq equation in [17]

\[ u t t + u u x + (u x)^2 + u u x x x = 0 \]

2- the Burgers equation in [7]

\[ u t + u u x = u x x = 0 \]

3- Kolmogorov-petrovskii-piskunov equation in [17]

\[ u t = u x x + (1-u)(u-a), \quad -1 < a < 1 \]

4- Nonlinear Wave equation in [14]

\[ u t t = u x x \]

5- Zabolotskaya-Khokhlov equation in [15]

\[ u x t - (u x x)(u y y) = 0 \]

Discussion:

Clearly the method of symmetry analysis of differential equations allows one to rigorously constrain the solution set of a particular problem, thereby simplifying it and facilitating the search for solution. Using the method developed by Lie, the equations are seemingly forced to reveal their symmetries. Obviously much more can be done using symmetry analysis than was demonstrated in the paper, however hopefully this glimpse will whet the reader’s appetite for more.

References:

تطبيق تناظر زمرة لي على المعادلات التفاضلية الجزئية اللاخطية
مها فالح جاسم
قسم أنظمة الحاسبات، المعهد التقني كركوك، جمهورية العراق

الملخص:
في هذا البحث قدمت طريقة تماثل لي الاعتيادية لحل المعادلات التفاضلية الجزئية اللاخطية. وتتميز هذه الطريقة بسهولتها ودقتها في حل المعادلات. في nghiênقات الأخيرة لتطبيق طريقة تمثال لي الاعتيادية لبعض المعادلات التفاضلية الجزئية اللاخطية.