FULL-ORDER OBSERVER DESIGN FOR NON-LINEAR DYNAMIC CONTROL SYSTEM

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Abstract

The main theme of this paper is to design a full-order non-linear dynamic observer for estimating the state space estimator from its non-linear input-output dynamic control system.

The quadratic Lyapunov function stabilization approach has been developed and the sufficient conditions for existence the dynamic observer for some class of non-linear input-output dynamic system have been presented and discussed. Illustrations are presented to demonstrate the validity of the of the presented procedure.

Introduction

Sometimes all state space variables not available for measurements or it is not practical to measure all of them, or it is too expensive to measure all state space variables.[5]

The concept of observability along with controllability first introduced by Kalman [2] plays an important role in both theoretical and practical concepts of modern control.

Observability of a system as conceptualized by Kalman fundamentally answers the question whether it is possible to identify any state \( x(t) \) at time \( t \) in the state space by observing the output \( y(t, t_i) \) to a given input \( u(t, t_i) \) over a finite time interval \( t_i - t, (t_i > t) \).

Luenberger observer [5] is essentially a full-order observer, the state variables of which give a one-to-one estimation of the linear dynamical system state variables.

In [1] the observer provides a smooth velocity estimate to be used by a trajectory tracking controller. The observer, controller and manipulator from a system where the observer error as well as the position and velocity tracking error tend to zero asymptotically.

In [7], the authors have consider a class of uncertain non-linear control system, and its controllability and stabilizability

\[
\dot{x}(t) = \left( \sum_{i=0}^{r} q_i A_i \right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + \sum_{i=0}^{r} f_i(x) \quad \ldots \quad (1)
\]

Where \( x(t) \) is n-state space, \( u(t) \) is m-control signals which belong to a class of piecewise continuous function.

\( f_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) to be continuous vector valued function, \( q_i \in \mathbb{R}^L \) is a vector of uncertain parameter, \( A_i \in \mathbb{R}^{n \times n} \), \( B_i \in \mathbb{R}^{n \times m} \), \( i = 0,1,\ldots,r \) are constant matrices.

Consequently [7], the Lyapunove function that stabilize a class of a non-linear system can be obtained easily from the result of nominal linear dynamic system and under some assumption both Lyapunove function of linear and non-linear dynamic system are obtained to be identical.

The work of this paper, focuses on the full order dynamic state observer for the uncertain non-linear control system

\[
\dot{x}(t) = \left( \sum_{i=0}^{r} q_i A_i \right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + \sum_{i=0}^{r} f_i(x) \\
y(t) = \sum_{i=1}^{r} c_i x_i + g(x) \quad \text{such that} \quad \sum_{i=1}^{r} c_i x_i = c^T x
\]

where \( x(t) \) is n-state vector, \( y(t) \) is n-output vector \( u(t) \) is m-control signals which belongs to a class of piecewise continuous function \( g(x), f_i(x) ; i = 1,2,\ldots,r \) are non-linear vector functions, respectively of dimensions \( n \) and \( p \)

\( A_i \in \mathbb{R}^{n \times n}, \quad B_i \in \mathbb{R}^{n \times m} \) and \( c^T \in \mathbb{R}^{p \times n} \); \( i = 0,1,\ldots,r \) are constant matrices.

The following definitions and theorems are needed later on.
\textbf{Definition}

The system
\[
\begin{align*}
\dot{x} &= Ax \\
y &= Cx
\end{align*}
\]
where \(x\) is \(n\)-state vector, \(y\) is \(m\)-output vector, \(A \in \mathbb{R}^{n \times n}\), \(C \in \mathbb{R}^{m \times n}\) is constant matrix, is said to be observable if every state \(x(t)\) can be determined from the observation of \(y(t)\) over a finite time interval \(t_e \leq t \leq t_i\).

\textbf{Definition}

A (vector-valued) function \(f(x)\) is said to be Lipschitz function if there exists a constant \(L\) such that for all \(x_1, x_2 \in \mathbb{R}^n\), the following inequality holds:
\[
\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|
\]
In this case \(L\) is said to be the Lipschitz constant of \(f\).

\textbf{Theorem}

The system is described by
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]
Where \((A, C)\) is observable means the dynamic system is observability if and only if the observability matrix constant matrices is completely state observability if and only if the observability matrix \( [C^T, C^TA; \cdots ; C^T A^{n-1}] \) has full rank equal to \(n\) (\(T\) denoted to conjugate transpose matrix).

\textbf{Theorem}

If all eigenvalues of the system \(\dot{x} = Ax\), where \(A \in \mathbb{R}^{n \times n}\) is a constant matrix having negative real parts then the solution of this system is asymptotically stable (exponentially stable).

\textbf{Lemma}

Let \(A\) be a symmetric matrix and let \(\lambda_{\min}(A)\) and \(\lambda_{\max}(A)\) be the smallest and largest eigenvalue of \(A\), respectively, then
\[
\lambda_{\min}(A)\|x\| \leq x^T A x \leq \lambda_{\max}(A)\|x\|, \quad \forall x \in \mathbb{R}^n
\]
where \(\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\), \(x_i\) is the \(i\)-the component of \(x\).

\textbf{Remark}

The Euclidean norm of matrix can be defined as:
\[
\|A\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}
\]
where \(|a_{ij}|\) is the absolute value of the matrix coefficient \(a_{ij}\).

\textbf{Problem Formulation}

Consider the non-linear dynamical control system described as follows:
\[
\begin{align*}
\dot{x}(t) &= \left(\sum_{i=1}^{m} q_i A_i\right) x(t) + \sum_{i=1}^{m} q_i B_i u_i(t) + \sum_{i=1}^{m} f_i(x) \\
y(t) &= \sum_{i=1}^{m} c_i x_i + g(x) \quad \text{such that} \sum_{i=1}^{m} c_i x_i = e^T x
\end{align*}
\]
\[
(x(0) = x_0)
\]

Where \(x(t) \in \mathbb{R}^n\) is unmeasurable state vector (see introduction \([5]\)), \(u(t)\) is the control input and \(y(t) \in \mathbb{R}^n\) is the output vector.

Suppose that the matrices \(A, B\) and \(C^T\) are constant matrices \(g(x(t)), f(x(t))\), \(i = 1, 2, \ldots, r\) are non-linear vector functions, respectively of dimensions \(n\) and \(p\).

Then a dynamical state observer of non-linear dynamical control system \((2)\) is constructed as follows:
\[
\begin{align*}
\dot{\hat{x}}(t) &= \left(\sum_{i=1}^{m} q_i A_i\right) \hat{x}(t) + \sum_{i=1}^{m} q_i B_i u_i(t) + \sum_{i=1}^{m} f_i(\hat{x}) + k_i \left[y(t) - c_i^T \hat{x} - g(\hat{x})\right] \\
\end{align*}
\]
\[
(3)
\]

Where the observed state is denoted by \(\hat{x}(t)\), and \(k_i\) is the observer gain matrix.

Define \(e(t) = x(t) - \hat{x}(t)\) \(\quad (4)\)
\(e(t)\) is the dynamical error between the actual state and state observer \(\hat{x}(t)\).

The dynamical error in state observer \((4)\) of non-linear dynamical control system \((2), (3)\)) has the following dynamic equation:
\[
\begin{align*}
\dot{e}(t) &= \left[\sum_{i=1}^{m} q_i A_i\right] e(t) - k_i c_i e(t) + \sum_{i=1}^{m} f_i(\hat{x}) - f_i(e(t)) + k_i [y(t) - g(\hat{x}) - g(e(t))]
\end{align*}
\]
\[
(5)
\]
and as the following the linear part of equation \((5)\)
\[
\begin{align*}
\dot{e}(t) &= \left[\sum_{i=1}^{m} q_i A_i\right] e(t) - k_i c_i e(t) \quad (6)
\end{align*}
\]
\[
e(0) = x(0) - \hat{x}(0) \quad (7)
\]

If the dynamic behavior of dynamical error \((5)\) is a symaptically stable, then the dynamical error \((5)\) will tend to zero with an adequate speed as the time tend to infinite then
the state $x(t)$ given in (2) will converge to the state observer $\hat{x}(t)$.

**Lemma**

Consider the linear dynamical control system

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i \right)x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) \quad \text{.........(8)}$$

$$y(t) = c^T x(t)$$

satisfied the following conditions $\left(\sum_{i=0}^{r} q_i A_i , c^T \right)$ is observable matrices and $q_i = 1, q_i > 0, i = 1, 2, \ldots, r$ are arbitrary, and let as the dynamical observer by

$$\dot{\hat{x}} = \left(\sum_{i=0}^{r} q_i A_i \right)\hat{x}(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + k_i \left[ y(t) - c^T \hat{x} \right] \quad \text{.........(9)}$$

Then the state observer has the dynamical observer state exponentially stabilization.

**Proof:**

Let $e(t) = x(t) - \hat{x}(t)$ from equations (8),(9) we have

$$\dot{e}(t) = \left[ \sum_{i=0}^{r} q_i A_i \right] - k_c c^T e(t) \quad \text{.........(10)}$$

with initial condition $e(0) = e_0$ has the solution

$$e(t) = e_0 \exp\left[ \sum_{i=0}^{r} q_i A_i \right] t$$

Due to the observability conditions of system (8) the matrix

$$\left[ \sum_{i=0}^{r} q_i A_i \right] - k_c c^T$$

is Hurwitz stable matrix

$$\|e(t)\| = \|e_0\| \|\exp[\sum_{i=0}^{r} q_i A_i - k_c c^T] t\|$$

where $\|\cdot\|$ is Euclidean norm and $\|\cdot\|_2$ is stable matrix norm

since $\left[ \sum_{i=0}^{r} q_i A_i \right] - k_c c^T$ is asymptotically stable thus there exist positive numbers $\alpha , \mu_0$ such that

$$\|e(t)\| \leq \mu_0 \|e_0\| \exp[-\alpha t]$$

$$\Rightarrow \|e(t)\| \leq \mu_0 \exp[-\alpha t]$$

where $\mu = \mu_0 \|e_0\|$ since $\alpha, \mu$ are positive numbers

Then $\|e(t)\| \to 0$ as $t \to \infty$ and hence $\|e(t)\| \to 0$ (exponentially)

$\Rightarrow x(t) \to \hat{x}(t)$ (exponentially stabilizable)

i.e $x(t) \converges$ as $t \to \infty$.

**Theorem**

Consider the linear dynamical system (8)

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i \right)x(t) + \sum_{i=0}^{r} q_i B_i u_i(t)$$

$$y(t) = \sum_{i=0}^{r} c_i x_i$$

$$x(0) = x_0$$

where $x \in R^n, u \in R^m, y \in R^p, A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times m}$ and the state variables are not available for measurement.

Consider the observer of linear dynamical control system (9)

$$\dot{\hat{x}}(t) = \left(\sum_{i=0}^{r} q_i A_i \right)\hat{x}(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + k_i \left[ y(t) - c^T \hat{x} \right]$$

$$\hat{x}(0) = \hat{x}_0$$

satisfied the following conditions

1. $\left[ \sum_{i=0}^{r} q_i A_i \right] c^T$ of linear dynamical system (8) is completely state observer

2. The uncertain condition $q_i = 1, q_i > 0, i = 1, 2, \ldots, r$ are arbitrary

3. The observer gain $k_c$ selected such that

$$\left[ \sum_{i=0}^{r} q_i A_i \right] c^T$$

is asymptotically stable matrix

4. The Riccati equation

$$\left[ \sum_{i=0}^{r} q_i A_i \right] c^T P + P \left[ \sum_{i=0}^{r} q_i A_i \right] c^T = -Q$$

as a unique positive definite solution $P$ for arbitrary positive definite matrix $Q$

5. assuming the Lyapunove function $V(x - \hat{x}) = (x - \hat{x})^T P (x - \hat{x})$ . Then the dynamical error is asymptotically stable via observer gain $K_e$

**Proof:**

Consider the linear dynamical system (8)

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i \right)x(t) + \sum_{i=0}^{r} q_i B_i u_i(t)$$

$$y(t) = \sum_{i=0}^{r} c_i x_i$$

such that $C^T x = \sum_{i=0}^{r} c_i x_i$

$$x(0) = x_0$$

the state observer of linear dynamical control system (8) is given as follows (9)

$$\dot{\hat{x}}(t) = \left(\sum_{i=0}^{r} q_i A_i \right)\hat{x}(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + k_i \left[ y(t) - c^T \hat{x} \right]$$
\[
\dot{x}(0) = \dot{x}_0
\]

Let \( e(t) = x(t) - \dot{x}(t) \)

We have dynamical error in state observer (9) of linear dynamical control system (8) is obtained to subtract (8) from (9) as follows

\[
\dot{e}(t) = \left[ \sum_{i=1}^{n} q_i \dot{A}_i - k_c c^T \right] e(t) \quad \text{..........................(10)}
\]

\[e(0) = e_0\]

To examine the stability of \( e(t) \), we consider the following quadratic Lyapunov function

\[V(e(t)) = e^T(t) Pe(t)\]

thus

\[V(e) = \left[ \left( \sum_{i=1}^{n} q_i A_i - k_c c^T \right) e \right]^T Pe + e^T P \left[ \left( \sum_{i=1}^{n} q_i A_i - k_c c^T \right) e \right]\]

\[V(e) = e^T \left[ \left( \sum_{i=1}^{n} q_i A_i - k_c c^T \right) + P + P \left( \sum_{i=1}^{n} q_i A_i - k_c c^T \right) \right] e\]

\[\text{since} \quad \left( \sum_{i=1}^{n} q_i A_i, c^T \right) \text{ is observable matrices and hence} \quad \left[ \sum_{i=1}^{n} q_i A_i - k_c c^T \right] \text{ stabilizable (see Remark (2) \cite{2})}\]

hence the following equation has a unique positive definite solution \( P \) such that

\[\left[ \sum_{i=1}^{n} q_i A_i - k_c c^T \right] P + P \left[ \sum_{i=1}^{n} q_i A_i - k_c c^T \right] = -Q \quad \text{..............(7)}\]

Thus from equation (6)

\[V(e) = -e^T Q e : Q \text{ is positive definite solution and then } V(e) < 0 , \forall e, t\]

since

\[V(e) = e^T Pe > 0 , \quad (P = P^T > 0) \text{ and } V(e) < 0\]

and of \( V(e) = e^T Pe \rightarrow 0 \)

\[
\Rightarrow \text{ The error dynamic system (10) is asymptotically stable via observer gain } k_c.
\]

Thus \( x(t) \approx \dot{x}(t) \) as \( t \rightarrow \infty \).

**Theorem**

Consider the linear dynamical system (2)

\[
\dot{x}(t) = \left( \sum_{i=1}^{n} q_i A_i \right) x(t) + \sum_{i=1}^{m} q_i B_i u_i(t) + \sum_{i=1}^{r} f_i(x)
\]

\[y(t) = C^T x + g(x) \quad \text{suchthat} \quad C^T x = \sum_{j=1}^{n} c_j x_j\]

\[x(0) = x_0\]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, \ y \in \mathbb{R}^p, \ A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ C \in \mathbb{R}^{p \times n}, \ f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \ g : \mathbb{R}^n \rightarrow \mathbb{R}^n \)

Consider the observer of linear dynamical control system (2) is given in equation (3)

\[
\dot{\hat{x}}(t) = \left[ \sum_{i=1}^{n} q_i \hat{A}_i \right] \hat{x}(t) + \sum_{i=1}^{m} q_i \hat{B}_i u_i(t) + k \left[ y(t) - c^T \hat{x} - g(\hat{x}) \right]
\]

\[\hat{x}(0) = \hat{x}_0\]

satisfied all the conditions (1,2,3,4 and 5) in theorem (2) and two conditions as the following

The non-linearity function

\[f_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n ; \ i = 1,2,\ldots,r \quad \text{and} \quad g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{are assumed to be Lipschitz condition with Lipschitz constant} \]

\[L_i ; \ i = 1,2,\ldots,r \quad \text{and respectively}
\]

\[\text{i.e.,} \quad \sum_{i=1}^{r} \| f_i(x) - f_i(\hat{x}) \| \leq \sum_{i=1}^{r} L_i \| x - \hat{x} \| \quad , \quad L_i > 0
\]

and \( \| g(x) - g(\hat{x}) \| \leq L \| x - \hat{x} \| \quad , \quad L > 0 \), \( \hat{x} \in \mathbb{R}^n \)

If

\[\left[ \sum_{i=1}^{n} L_i - L \right] \leq -\frac{k_{\text{min}}(Q)}{2k_{\text{min}}(P)}
\]

Then the dynamical error (5)

\[
\dot{e}(t) = \left[ \sum_{i=1}^{n} q_i A_i - k_c c^T \right] e(t) + \sum_{i=1}^{r} \left[ f_i(x) - f_i(\hat{x}) \right] - k \left[ g(x) - g(\hat{x}) \right]
\]

is asymptotically stable via observer gain parameter \( k_c \).

**Proof:**

The linear dynamical system (2)

\[
\dot{x}(t) = \left( \sum_{i=1}^{n} q_i A_i \right) x(t) + \sum_{i=1}^{m} q_i B_i u_i(t) + \sum_{i=1}^{r} f_i(x)
\]

\[y(t) = C^T x + g(x)
\]

\[x(0) = x_0\]

The state observer of non-linear dynamical control system (2) is given as follows (3)

\[
\dot{\hat{x}}(t) = \left[ \sum_{i=1}^{n} q_i \hat{A}_i \right] \hat{x}(t) + \sum_{i=1}^{m} q_i \hat{B}_i u_i(t) + \sum_{i=1}^{r} f_i(x) + \sum_{i=1}^{r} f_i(x) + k \left[ y(t) - x - C^T \hat{x} - g(\hat{x}) \right]
\]

\[x(0) = x_0\]

The dynamical error in state observer (3) of non-linear linear dynamical control system is obtained to subtract (2) from (3) as follows:

\[
\dot{e}(t) = \left[ \sum_{i=1}^{n} q_i A_i - k_c c^T \right] e + \sum_{i=1}^{r} \left[ f_i(x) - f_i(\hat{x}) \right] - k \left[ g(x) - g(\hat{x}) \right]
\]
Let \( e(t) = x(t) - \hat{x}(t) \)

Now, we consider the following quadratic Lyapunov function.

For examining the stability of \( e(t) \)

Let \( V(e) = e^T P e \quad V(0) = 0 \) , \( V(e) > 0 \)

And \( \dot{V}(e) = e^T P e + e^T P \hat{e} \)

\[
\dot{V}(e) = \left[ \sum_{i=1}^{r} q_{i} A_{i} - k_{i} c_{i} \right] e(t) + \sum_{i=1}^{r} \left[ f_{i}(x) - f_{i}(\hat{x}) \right] - k_{i} \left[ g_{i}(x) - g_{i}(\hat{x}) \right] \right] P e + e^T P \left[ \sum_{i=1}^{r} q_{i} A_{i} - k_{i} c_{i} \right] e(t) + \sum_{i=1}^{r} \left[ f_{i}(x) - f_{i}(\hat{x}) \right] - k_{i} \left[ g_{i}(x) - g_{i}(\hat{x}) \right] \right] \]

\[
\Rightarrow V(e) = -e^T Q e + 2 P \sum_{i=1}^{r} L_{i} \left[ \sum_{i=1}^{r} L_{i} \right] - 2 L \left[ \sum_{i=1}^{r} L_{i} \right] \]

\[
\dot{V}(e) \leq -\lambda_{\text{min}}(Q) e^T e + 2 \left[ \sum_{i=1}^{r} L_{i} \right] - 2 L \left[ \sum_{i=1}^{r} L_{i} \right] \]

\[
\Rightarrow \dot{V}(e) \leq -\lambda_{\text{min}}(Q) e^T e + 2 \left[ \sum_{i=1}^{r} L_{i} \right] - 2 L \left[ \sum_{i=1}^{r} L_{i} \right] \leq 0 \]

So, we have that \( e(t) \rightarrow 0 \) as \( t \rightarrow \infty \) as \( x(t) \equiv \hat{x}(t) \) Thus

\section*{Algorithm}

Step(1). Consider the linear and non-linear dynamical system (8)-(2) with respectively.

Step(2). Check the pair \( \left\{ \sum_{i=1}^{r} q_{i} A_{i}, c_{i} \right\} \) is observable

Step(3). Check the following Lipschitz conditions

\[
\sum_{i=1}^{r} \left\| f_{i}(x) - f_{i}(\hat{x}) \right\| \leq L \| x - \hat{x} \| \]

\[
\| g_{i}(x) - g_{i}(\hat{x}) \| \leq L \| x - \hat{x} \| \quad \text{for} \quad t \in R^n
\]

And design the observer dynamic by (9)-(3)

Step(4). Select \( k_{e} \) that makes

\[
\left[ \sum_{i=1}^{r} q_{i} A_{i} - k_{i} c_{i} \right] \text{ asymptotically stable by using pole placement (see[3])}
\]

Step(5). Let the dynamic error

\[
e(t) = x(t) - \hat{x}(t) \quad \text{and} \quad e^T (0) = e_0
\]

\[
e(t) = \left[ \sum_{i=1}^{r} q_{i} A_{i} - k_{e} c_{i} \right] e(t) + \sum_{i=1}^{r} \left[ f_{i}(x) - f_{i}(\hat{x}) \right] - k_{e} \left[ g_{i}(x) - g_{i}(\hat{x}) \right]
\]

\[
e(0) = e_0
\]

Step(6). Set \( V(t) = V(e(t)) = e^T (t) P e(t) k_{e} \)

where \( P \) is the unique positive definite solution of

\[
\left[ \sum_{i=1}^{r} q_{i} A_{i} - k_{i} c_{i} \right] P + P \left[ \sum_{i=1}^{r} q_{i} A_{i} - k_{i} c_{i} \right] = Q
\]

for arbitrary positive definite matrix \( Q \)
Step(7). Check \( \left( \sum_{i=0}^{\infty} L_i - k_i L \right) \leq \frac{\lambda_{\text{max}}(Q) - \lambda_{\text{min}}(P)}{2 \lambda_{\text{max}}(P)} \), where \( \sum_{i=0}^{\infty} L_i \), \( L \) and \( k_i \) are found in step(3), step(4) with respectively and \( \lambda_{\text{min}}(Q) \) denotes the smallest eigenvalue of \( Q \), \( \lambda_{\text{max}}(P) \) denotes the largest eigenvalue of \( P \).

**Illustration (1)**

Consider the linear dynamical control observation error description

\[
\dot{e}(t) = \left( \sum_{i=0}^{\infty} q_i A_i \right) - k \cdot e^T \begin{bmatrix} c \end{bmatrix} e(t)
\]

where \( q_i = 1 \), \( q_i > 0 \); for \( n = 2 \) gives the matrices

\[
A_1 = \begin{bmatrix} -3 & 0 \\ -2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}
\]

\[
c^T = [1 \ 1],
\]

where \( q_1 = 1 \), \( q_2 = 1 \), \( q_2 = 2 \); where

\[
\sum_{i=0}^{\infty} q_i A_i = q_1 A_1 + q_2 A_2 = A_1 + A_2 + 2 A_3 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}
\]

Let us examine state observability of the

\[
\left[ \sum_{i=0}^{\infty} q_i A_i , c^T \right], \text{ notice that}
\]

\[
\text{rank} \left[ c^T \right] = \text{rank} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = 2
\]

\[
\Rightarrow \text{the vectors } \left[ c^T \right], A^T \left[ c^T \right] \text{ are independent and the rank of the matrix}
\]

\[
\left[ c^T : A^T \left[ c^T \right] \right] \text{ is two}
\]

\[
\Rightarrow \left[ \sum_{i=0}^{\infty} q_i A_i , c^T \right] \text{ is observable}.
\]

Then, the system is completely state observable

\[
\Rightarrow \left[ \sum_{i=0}^{\infty} q_i A_i \right] - k \cdot c^T \text{ is asymptotically stable}
\]

(see problem formulation)

Now, by using the formula

\[
\left[ \sum_{i=0}^{\infty} q_i A_i \right] - k \cdot c^T = -E_{2x2} \text{ such that } E_{2x2} \text{ is arbitrary diagonal matrix}
\]

\[
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}
\]

Now, we illustrate the proceeding main theorem (2) and we find Linear Dynamical control observation error (5) is asymptotically stabilizable.

\[
\Rightarrow \text{V(e)} \text{ positive defined function and have the system (10) is stabilized by using Lyapunove } V(e) \text{ and observer gain } k.
\]

In example (1) we apply the proceeding main theorem (2) and we find Linear Dynamical control observation error (5) is asymptotically stabilizable.

Now, we illustrate the proceeding main theorem (3), we consider the non-linear dynamical control observation error.

**Illustration (2)**

where \( g(x) \), \( f(i,x) \) for \( i = 1, 2 \) are non-linear vector functions and
\[ A_i \in R^2, \quad c^T \in R^2; \quad i = 0,1,2 \]
and
\[
\left( \sum_{i=1}^{m} q_i A_i \right) - k, c^T = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}
\]
is stable matrix
\[
f_1(x) = \begin{bmatrix} 0.03 \sin x \\ 0.05 x \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} -0.2 \cos x \\ -0.02 \cos x \end{bmatrix}
\]
and \( g(x) = \frac{1}{5} \) are defined on the region
\[
D = \{ f, x \in R \mid \alpha \leq 1, \quad |x| \leq 0.01 \}
\]
\[
\left\| f_1(x) - f_1(\hat{x}) \right\| = \left\| 0.03 \sin x - 0.03 \sin \hat{x} \right\| + \left\| 0.05 x - 0.05 \hat{x} \right\|
\]
\[
\leq \left\| \dot{f}_1(x) \right\| \left\| x - \hat{x} \right\| + 0.05 \left\| x - \hat{x} \right\|
\]
\[
\leq \left( 0.03 \left\| \sin x \right\| + 0.03 \left\| \sin \hat{x} \right\| \right) \left\| x - \hat{x} \right\| + 0.05 \left\| x - \hat{x} \right\|
\]
\[
\leq 0.0803 \left\| x - \hat{x} \right\|
\]
\[
\Rightarrow \text{Lipschitz constant } L_1 = 0.0803
\]
\[
\left\| f_2(x) - f_2(\hat{x}) \right\| = \left\| -0.2 \cos x + 0.2 \cos \hat{x} \right\| + \left\| -0.02 \cos x + 0.02 \cos \hat{x} \right\|
\]
\[
\leq \left\| \dot{f}_2(x) \right\| \left\| x - \hat{x} \right\| + 0.02 \left\| x - \hat{x} \right\|
\]
\[
\leq 0.2 \left\| x - \hat{x} \right\| + 0.02 \left\| x - \hat{x} \right\|
\]
\[
\leq 0.22 \left\| x - \hat{x} \right\|
\]
\[
\Rightarrow \text{Lipschitz condition } L_2 = 0.223
\]
and \( g(x) \frac{1}{5} )
and \( k_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow k_i g(x) = \begin{bmatrix} \frac{1}{5} \mid \frac{1}{5} \end{bmatrix}
\]
\[
\Rightarrow \left\| k_i (g(x) - g(\hat{x})) \right\| = \left\| \frac{x}{5} - \frac{\hat{x}}{5} \right\| + \left\| \frac{x}{5} - \frac{\hat{x}}{5} \right\|
\]
\[
= \frac{2}{5} \left\| x - \hat{x} \right\|
\]
\[
\Rightarrow L = \frac{2}{5} = 0.4
\]
finally, we can discussed the condition
\[
\left( \sum_{i=1}^{m} L_i \right) - L < \frac{\lambda_{\max}(Q)}{2\lambda_{\max}(P)}
\]
as the following
since the matrix \( P = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.166666 \end{bmatrix} \) been found by solution
\[
\begin{bmatrix} -2 & 0 \end{bmatrix}^T P + P \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = -Q, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
we have \( \lambda_{\max}(P) = 0.25 \) and \( \lambda_{\max}(Q) = 1 \)
since \( L_1 = 0.0803, \quad L_2 = 0.22, \quad L = 0.4 \)
\[
\left( \sum_{i=1}^{m} L_i \right) - L = -0.0997 < \frac{\lambda_{\max}(Q)}{2\lambda_{\max}(P)} = 2
\]
the condition
\[
\left( \sum_{i=1}^{m} L_i \right) - L < \frac{\lambda_{\max}(Q)}{2\lambda_{\max}(P)}
\]
is satisfied
the system (4) is asymptotically stable in the large using the Lyapunove function
\[
V(x - \hat{x}) = 0.25 e_1^2 + 0.1666666 e_2^2
\]
\[
\Rightarrow V(x - \hat{x}) = 0.25 e_1^2 + 0.1666666 e_2^2
\]

**Conclusion**

- Sufficient conditions were given for the design observers for a class of non-linear system
- This system is characterized by non-linear functions which are Lipschitz in nature.

**References**


الخلاصة

ان الهدف الرئيسي من هذا البحث هو تصميم مخمن دينامي كامل الرتبة غير خطئي لتخمين فضاء الحالة من (input-output) خلال نظام سيطرة دينامي مدخل-مخرج غير خطئي كما وتم تطوير استثناء دالة ليابانوف التربيعية للاستقرارية وتم عرض العديد من الشروط الكافية لوجودية النظام المخمن الدينامي غير الخطئي. قدمت كذلك امثلة توضيحية لتحديد دراسة صحة الاستنتاج المقدمة.