The Homotopy Analysis Method for Solving Multi- Fractional Order Integro-Differential Equations

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Abstract

In this paper we use the homotopy analysis method to solve special types of the initial value problems that consist of multi-fractional order integro differential equation in which the fractional derivative and fractional integral in them are described in the Caputo sense and Riemann-Liouville sense respectively. Numerical examples are solved by using this method. These examples shows that high accuracy, simplicity and efficiency of this method.

Keywords: homotopy analysis method, fractional integro differential equations, Caputo fractional derivative.

1. Introduction

In recent years, there has been a growing interest in the fractional integro-differential equations which play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electro statistics, [1].

The fractional integro-differential equations are usually difficult to solve analytically; so many different methods are used to obtain the solution of these equations [2] used the collocation method to solve the initial value problem that consists of the fractional integro differential equation of the form

$$D_*^{\alpha} y(t) = a(t)y(t) + f(t) + \int_0^t \mathrm{K}(t,s)F(y(s))ds \ , \quad t \in [0,1] \ \ldots\ldots\ldots\ldots\ldots (1)$$

together with the initial condition $y(0) = \beta$

[1] used Adomain decomposition method to solve the initial value problem given by eq. (1) and [3] used the homotopy analysis method to solve the initial value problem that given by eq.(1).

The homotopy analysis method (HAM) was first proposed by Liao in his Ph.D. thesis [4] in which he employed the basic ideas of the homotopy analysis in topology to propose a general analytic method for solving non-linear problems. The validity of HAM is independent of whether or not there exists small parameter in the considered equation. Besides, different from all previous numerical and analytical methods, it provides us with a simple way to adjust and control the convergence of solution series. This method has been successfully applied to solve many types of nonlinear problems [5]. [6], [7] and [3].

In this work the homotopy analysis method has been used to solve the initial value problem that consists of the multi-fractional order integro-differential equation of the form:

$$D_*^{\alpha} y(t) = P(t)\, y(t) + g(t) + J^{\beta} F(y(t)) \ ,$$
$$n-1 < \alpha \le n \ , \ k-1 < \beta \le k, \ n,k \in N \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.a)$$

together with the initial condition

$$y(0) = \eta \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.b)$$

Some illustrative examples are presented to show the efficiency of the present method for our problem in comparison with the exact solution.

2. The Homotopy Analysis Method [2]

In this section, we give some basic concepts of the homotopy analysis method. To do this, consider

$$N[y(t)] = 0 \ , t \ge 0$$

together with the initial condition:

$$y(0) = \eta$$

where N is a nonlinear operator, t is the independent variable and y is the unknown function. By means of generalizing the traditional homotopy method constructed the so called zero-order deformation equation:

$$(1-q)L[\phi(t,q) - y_0(t)] = qhH(t)N[\phi(t,q)] \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is non-zero auxiliary parameter,

$H(t) \neq 0$ is a non-zero auxiliary function and L is an auxiliary linear operator with the following property $L[y(t)] = 0$ when y(t)=0, $y_0(t)$is the initial guess of the exact solution y(t). It should be emphasized that one has great freedom to choose the initial guess $y_0(t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h and the auxiliary function $H(t)$.

When q=0, the zero-order deformation eq.(3) becomes:

$$\phi(t,0) = y_0(t) \quad \text{.......} (4)$$

and when q=1, since $h \neq 0$ and $H(t) \neq 0$, the zero-order deformation eq.(3) is equivalent to:

$$\phi(t,1) = y(t) \quad \text{.......} (5)$$

Thus, as q increases from 0 to 1, the solution $\phi(t,q)$ varies continuously from the initial guesses $y_0(t)$ to the exact solution $y(t)$. Such a kind of continuous variation is called deformation in homotopy. Expanding $\phi(t,q)$ in a Taylor series with respect to q, we have:

$$\phi(t,q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \cdot q^m \quad \text{......} (6)$$

where

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t,q)}{\partial q^m}\bigg|_{q=0}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series eq.(6) converges at q=1, then we have under these assumption the solution series becomes:

$$y(t) = \phi(t,1) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \quad \text{.......} (7)$$

Define the vector

$$\vec{y_n}(t) = \{y_0(t), y_1(t), y_2(t), \dots, y_n(t)\}$$

By differentiating the zero-order deformation eq.(3) m times with respect to the embedding parameter q and then setting q=0 and finally dividing them by m!, we obtain at the mth order deformation equation:

$$L[y_m(t) - x_m y_{m-1}(t)] = hH(t)R_m(\vec{y}_{m-1}(t)) \quad \text{.......} (8)$$

where:

$$R_m(\vec{y}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t,q)]}{\partial q^{m-1}}\bigg|_{q=0} \quad \text{.......} (9)$$

and $x_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$

3. The Homotopy Analysis Method for Solving Eq.(2)

In this section we use the homotopy analysis method for solving the initial value problem given by eq. (2). To do this, we rewrite eq. (2.a) in the form

$$N[y(t)] = 0$$

where

$$N[y(t)] = D_*^{\alpha} y(t) - p(t)y(t) - g(t) - J^{\beta}F(y(t))$$

The corresponding mth order deformation given by eq.(8) reads as:

$$L[y_m(t) - x_m\, y_{m-1}(t)] = hH(t)\, R_m[\vec{y}_{m-1}(t)]$$

$$y_m(0) = 0 \quad \text{.......} (10)$$

where

$$R_m(\vec{y}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left(D_*^{\alpha}\phi(t,q) - p(t)\phi(t,q) - g(t) - J^{\beta}[F(\phi(t,q))]\right)\bigg|_{q=0} \quad \text{.......} (11)$$

The corresponding homotopy-series solution is given by

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \quad \text{.......} (12)$$

It is worth to present a simple iterative scheme for $y_m(t)$. To this end, taken a nonzero auxiliary function $H(t)$=l and the linear operator $L = D_*^{\alpha}$,

$(n-1 < \alpha \leq n, n \in N)$. This is

substituted into eq.(10) to give the recurrence relation

$$y_0(t) = \eta \quad \text{.......} (13)$$

$$D_*^{\alpha} y_m(t) = x_m\; D_*^{\alpha}\, y_{m-1}(t) + h R_m[\vec{y}_{m-1}(t)] \quad \text{.......} (14)$$

By applying the Riemann-Liouville integral operator J^{α} on both sides of eq.(14), we have

$$y_m(t) = x_m\; y_{m-1}(t) - x_m \sum_{j=0}^{n-1} y_{m-1}^{(j)}(0^+)\frac{t^j}{j!} +$$

$$hJ^{\alpha}[R_m(\vec{y}_{m-1}(t))] \quad ,m = 1,2,3,\dots \quad \text{......} (15)$$

By evaluating eq. (15) at each m = 1,2,3,…, we can get $y_1(t)$, $y_2(t)$,… these functions together with $y_0(t)$ are substituted into eq. (12) to get the approximated solution of the initial value problem given by eq. (2).

4. Illustrative Examples

In this section we give two examples to show the efficiency of the homotopy analysis method for solving the initial value problem given by eq. (2).

Example (1):

Consider the initial value problem that consists of the multi-fractional order integro-differential equation

$$D_*^{0.5} y(t) = \frac{6}{\Gamma(3.5)} t^{2.5} - \frac{\Gamma(4)}{\Gamma(5.5)} t^{4.5} + J^{1.5} y(t)\ ,\ t \in [0,1] \ \ldots\ldots (16)$$

together with the initial condition $y(0) = 0$

The initial value problem is constructed such that the exact solution of it is $y(t) = t^3$.

We use the homotopy analysis method to solve the initial value problem. To do this, we begin with $y_0(t) = 0$, and consider eq.(11) we can construct the homotopy as follows:

$$R_m(\vec{y}_{m-1}(t)) = D_*^{0.5} y_{m-1}(t) - J^{1.5} y_{m-1}(t) - (1 - x_m)\left[\frac{6}{\Gamma(3.5)} t^{2.5} - \frac{\Gamma(4)}{\Gamma(5.5)} t^{4.5}\right]$$

and the m[th]-order deformation equations for $m \geq 1$ becomes:

$$y_m(t) = x_m\ y_{m-1}(t) - x_m \sum_{j=0}^{0} y_{m-1}^{(j)}(0^+) \frac{t^j}{j!} + h\, J^{0.5}[R_m(\vec{y}_{m-1}(t))]$$

for $m = 1$, we have

$$y_1(t) = \frac{-h}{\Gamma(0.5)} \int_0^t (t-\tau)^{-0.5} \left[\frac{6}{\Gamma(3.5)} \tau^{2.5} - \frac{\Gamma(4)}{\Gamma(5.5)} \tau^{4.5}\right] d\tau$$

$$y_1(t) = -0.5642h(1.7725t^3 - 8.8623 \times 10^{-2} t^5)$$

and for $m = 2,3,\ldots$

$$y_m(t) = (1+h)\, y_{m-1}(t) - (1+h)\, y_{m-1}(0) - \frac{h}{\Gamma(2)} \int_0^t (t-\tau)\ y_{m-1}(\tau)\ d\tau$$

By evaluating the above equation for m=2, one can have

$$y_2(t) = (1+h)\, y_1(t) - \frac{h}{\Gamma(2)} \int_0^t (t-\tau)\, y_1(\tau)\, d\tau$$

$$y_2(t) = (-0.5642 - 0.5642\, h)\, h\, (1.7725 t^3 - 8.8623 \times 10^{-2} t^5) - h(1.1905 \times 10^{-3} h t^7 - 5 \times 10^{-2} h t^5)$$

By evaluating the above equation for m=3, one can have

$$y_3(t) = (1+h)\, y_2(t) - \frac{h}{\Gamma(2)} \int_0^t (t-\tau)\, y_2(\tau)\, d\tau$$

$$y_3(t) = (1+h)[(-0.5642 - 0.5642\, h)\, h\, (1.7725 t^3 - 8.8623 \times 10^{-2} t^5) - h(1.1905 \times 10^{-3} h t^7 - 5 \times 10^{-2} h t^5)] - h(-1.6534 \times 10^{-5} h^2 t^9 + 1.1905 \times 10^{-3} h t^7 + 2.3809 \times 10^{-3} t^7 h^2 - 5 \times 10^{-2} h t^5 - 5 \times 10^{-2} h^2 t^5)$$

By continuing in this manner one can get the approximated solution $y^*(t) = \sum_{m=0}^{10} y_m(t)$ The following table gives the absolute error between the exact solution $y(t)=t^3$ and the approximated solution $y^*(t) = \sum_{m=0}^{10} y_m(t)$ at some specific points for different values of h.

Table (1)
Numerical results of Example (1).

t	h= -0.7	h= -0.9	h= -1	h= -1.1	h= -1.3
0	0	0	0	0	0
0.1	5.97396×10^{-9}	1.0454×10^{-13}	0	9.4565×10^{-14}	5.7776×10^{-9}
0.2	4.9466×10^{-8}	9.4963×10^{-13}	1×10^{-22}	6.3216×10^{-13}	4.3221×10^{-8}
0.3	1.7655×10^{-7}	3.8913×10^{-12}	0	1.5001×10^{-12}	1.2962×10^{-7}
0.4	4.5132×10^{-7}	1.1758×10^{-11}	0	1.7627×10^{-12}	2.5639×10^{-7}
0.5	9.6714×10^{-7}	3.0245×10^{-11}	0	1.1702×10^{-12}	3.8288×10^{-7}
0.6	1.8609×10^{-6}	7.0274×10^{-11}	1×10^{-20}	5.5209×10^{-12}	4.3751×10^{-7}
0.7	3.3312×10^{-6}	1.5193×10^{-10}	0	1.4535×10^{-11}	3.2709×10^{-7}
0.8	5.6624×10^{-6}	3.1093×10^{-10}	2×10^{-20}	2.4768×10^{-11}	4.5723×10^{-7}
0.9	9.2558×10^{-6}	6.0908×10^{-10}	1×10^{-20}	3.0600×10^{-11}	7.5819×10^{-7}
1	1.4671×10^{-5}	1.1508×10^{-9}	0	2.3401×10^{-11}	1.8388×10^{-6}

From the above table one can deduce that for h = -1 one can get the best solution of the initial value problem given by eq.(16).

Example (2):

Consider the initial value problem that consists of the multi-fractional order nonlinear integro-differential equation

$$D_*^{0.75}y(t)=\frac{1}{\Gamma(1.25)}t^{0.25}-\frac{2}{\Gamma(3.75)}t^{2.75}+ J^{0.75}y^2(t),\ t\in[0,1]\ \text{.............}\ (17)$$

together with the initial condition $y(0)=0$

The initial value problem is constructed such that the exact solution of it is $y(t)=t$.

We use the homotopy analysis method to solve the initial value problem. To do this, we begin with $y_0(t)$ = 0, and consider eq.(11) we can construct the homotopy as follows:

$$R_m(\vec{y}_{m-1}(t))=D_*^{0.75}y_{m-1}(t)-J^{0.75}\left[\sum_{i=0}^{m-1}y_i\,y_{m-1-i}\right]-(1-x_m)\left[\frac{1}{\Gamma(1.25)}t^{0.25}-\frac{2}{\Gamma(3.75)}t^{2.75}\right]$$

and the m[th]- order deformation equations for $m\geq 1$ becomes:

$$y_m(t)=x_m\ y_{m-1}(t)-x_m\sum_{j=0}^{0}y_{m-1}^{(j)}(0^+)\frac{t^j}{j!}+ h\,J^{0.75}[R_m(\vec{y}_{m-1}(t))]$$

for m = 1, we have

$$y_1(t)=\frac{-h}{\Gamma(0.75)}\int_0^t(t-\tau)^{-0.25}\left[\frac{1}{\Gamma(1.25)}\tau^{0.25}-\frac{2}{\Gamma(3.75)}\tau^{2.75}\right]d\tau$$

$$y_1(t)=-h(t-0.1719t^{3.5})$$

and for $m=2,3,\ldots$

$$y_m(t)=(1+h)\,y_{m-1}(t)-(1+h)\,y_{m-1}(0)-\frac{h}{\Gamma(1.5)}\int_0^t(t-\tau)^{0.5}\left[\sum_{i=0}^{m-1}y_i(\tau)\,y_{m-1-i}(\tau)\right]d\tau$$

By evaluating the above equation for m=2, one can have

$$y_2(t)=(1+h)\,y_1(t)-(1+h)y_1(0)$$

$$y_2(t)=-(1+h)\,h(t-0.1719t^{3.5})$$

By evaluating the above equation for m=3, one can have

$$y_3(t)=(1+h)\,y_2(t)-(1+h)\,y_2(0)-\frac{h}{\Gamma(1.5)}\int_0^t(t-\tau)^{0.5}\ y_1^2(\tau)\ d\tau$$

$$y_3(t)=-(1+h)(-1-h)\,h(t-0.1719t^{3.5})-1.1284h(-2.2156\times10^{-2}h^2t^6+0.1524h^2t^{3.5}+1.1069\times10^{-3}h^2t^{8.5})$$

By continuing in this manner one can get the approximated solution $y^*(t)=\sum_{m=0}^{9}y_m(t)$ The following table gives the absolute error between the exact solution $y(t)=t$ and the approximated solution $y^*(t)=\sum_{m=0}^{9}y_m(t)$ at some specific points for different values of h.

Table (2)

Numerical results of Example (2).

t	h= -0.9	h= -1	h= -1.1	h= -1.2	h= -1.3
0	0	0	0	0	0
0.1	2.8571×10^{-10}	1.7380×10^{-18}	3.7616×10^{-10}	8.7649×10^{-8}	2.6686×10^{-6}
0.2	3.668×10^{-9}	2.0082×10^{-14}	6.0937×10^{-9}	6.1819×10^{-7}	1.2859×10^{-5}
0.3	3.0756×10^{-8}	4.7601×10^{-12}	5.9012×10^{-8}	3.3046×10^{-6}	5.2101×10^{-5}
0.4	1.9376×10^{-7}	2.2925×10^{-10}	4.0551×10^{-7}	1.4324×10^{-5}	1.8043×10^{-4}
0.5	9.8324×10^{-7}	4.6002×10^{-9}	2.1535×10^{-6}	5.2949×10^{-5}	5.5255×10^{-4}
0.6	4.1974×10^{-6}	5.2947×10^{-8}	9.3315×10^{-6}	1.7184×10^{-4}	1.5305×10^{-3}
0.7	1.5542×10^{-5}	4.1445×10^{-7}	3.4257×10^{-5}	4.9942×10^{-4}	3.8900×10^{-3}
0.8	5.1054×10^{-5}	2.4419×10^{-6}	1.0943×10^{-4}	1.3192×10^{-3}	9.1663×10^{-3}
0.9	1.5138×10^{-4}	1.1559×10^{-5}	3.1019×10^{-4}	3.2032×10^{-3}	2.0182×10^{-2}
1	4.1063×10^{-4}	4.5966×10^{-5}	7.09148×10^{-4}	7.2138×10^{-3}	4.1775×10^{-2}

From the above table one can deduce that for h = -1 one can get the best solution of the initial value problem given by eq.(17).

5. Conclusion

The homotopy analysis method for solving the initial value problem of the multi-fractional order integro-differential equation given by eq.(2) gives a greater region of convergence with the exact solution, by choosing proper values for the auxiliary parameter h and by using a suitable auxiliary linear operator $L = D_*^{\alpha}$. Finally generally speaking the proposed approach can be used to solve other types of nonlinear problems for the fractional calculus field.

6. References

[1] R.C. Mittal and R. Nigam, "Solution of fractional integro-differential equations by Adomain decompositions method", international journal of applied mathematics and mechanics, Vol. 4, No. 2, pp.87-94, 2008.
[2] E. A. Rawashdeh, "Numerical solution of fractional integro-differential equations by collocations method", applied mathematics and computations, Vol. 176, No. 1, pp.1-6, 2005.
[3] O. H. Mohammed, "Solution of fractional integro-differential equations by homotopy analysis method'', Journal of Al-Nahrain university (Science), Vol. 13, No. 3, 2010.
[4] S.J. Liao, "The proposed homotopy analysis technique for the solution of nonlinear problems", Ph.D. thesis, Shanghai Jiao Tong university, 1992.
[5] H. Jafari and S. Seifi, "Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation", common nonlinear SciNamer. Simul., Vol. 14, No. 5, pp. 2006-2012, 2004.
[6] S. Momani and Z. Odibat, "Numerical approach to differential equations of fractional order", J. computer Appt. Match, Vol. 207, pp. 96-110, 2007.
[7] M. Zurigat, "Analytical approximate solutions of systems of fractional algebraic -differential equations by homotopy analysis method", computer and mathematics with applications, Vol. 59, pp.1227-1235, 2010.

الخلاصة

في هذا البحث استعملنا طريقة تحليل الهوموتوبي لحل انواع خاصة من مسائل القيم الابتدائية والتي تحتوي على المعادلات التفاضلية – التكاملية ذات الرتب الكسورية المزدوجة والتي تحتوي على المشتقة الكسورية من نوع كابوتو والتكامل الكسوري من نوع ريمان – ليوفيل. قمنا بحل بعض الامثلة العددية باستخدام هذه الطريقة. هذه الامثلة أضهرت دقة عالية، سهولة، كفاءة لهذه الطريقة.