On Some Properties of Maximal M-Open Sets

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Abstract
In this work we introduce maximal m-open set in minimal structure spaces and study some of their basic properties in these spaces.

1. Introduction
Recall that a subset U of a topological space X is said to be maximal open set if any open set which contains U is X or U. This concept is introduced by Nakaoka F. and Oda N. in [1]. Then they gave other equivalence definition for the maximal open set by lemma 2.2.1 [a subset U of a topological space X is maximal open set if any open set W, then U ∪ W = X or W ⊆ U]. Also if X is a non empty set, then by minimal structure space (X, mX) (or mX-space) means a collection mX of subsets of X such that X, ∅ ∈ mX, members of mX called m-open and their complements called m-closed and the m-closure (m-interior) of a subset A of X denoted by m-Cl(A) (m-Int(A)) is defined as: m-Cl(A) = ∩{U: U ∈ mX, A ⊆ U} [2]. The aim this work is to give the concept of maximal m-open set. Some theorems and properties relative to this concept are introduced.

2. Maximal m-Open Sets
In this section we give the definition of maximal m-open subset of minimal structure space (X, mX). Throughout this section, we assume (X, mX) to be minimal structure space.

Definition 2.1:
A proper nonempty m- open subset U of X is said to be a maximal m- open set if for any m-open subset W of X, then U ∪ W = X or W ⊆ U.

Lemma 2.2:
(i) If a subset U of X is maximal m-open, then any m-open set which contains U is X or U.
(ii) If U and V are maximal m-open sets, then either U = V or U ∪ V = X.

Proof:
(i) Suppose that there is an m-open set W of X such that U ⊆ W ⊆ X, then U ∪ W = W but U is maximal m-open, then U ∪ W = X so W = X which is a contradiction. Therefore any m-open which contains U is X or U.
(ii) It follows directly from the Definition 2.1.

Remark:
The converse of lemma 2.2.(i), is not true in general. For example if X={1,2,3,4,5} and mX={X,∅,{1,2,3},{4,5},{4}}, then any m-open set which contains {1,2,3} is X or {1,2,3}. But {1,2,3} is not maximal m-open set.

Definition 2.3:
Let (X, mX) be a minimal structure. A subset W of X is said to be an m-open neighborhood of x ∈ X, if there exists U ∈ mX and x ∈ U ⊆ W.

Remark:
In above definition if also W ∈ mX, then W is called m-open neighborhood.

Proposition 2.4:
Let U be a maximal m-open set. If x is an element of U, then for any m-open neighborhood W of x, W ∪ U = X or W ⊆ U.

Proof:
It follows from the fact that any m-open neighborhood is m-open.

Theorem 2.5:
Let U be a maximal m-open set. If x is an element of U, then for any m-open neighborhood W of x, W ∪ U = X or W ⊆ U.

Proof:
Suppose neither that U ≠ W nor V ≠ W. Since W is maximal m-open set then X−U and X−V are subsets of W and so
(X−U)∪(X−V)= X−(U∩V) ≤ W but U∩V sets, w≤W. Then we have W=X is a contradiction. Therefore U=W or V=W. ■

**Theorem 2.6:**
Let U, V, and W be maximal m-open which are different from each other. Then, U∩V≠U∩W. ■

**Proof:**
The proof follows directly from Theorem 2.5. ■

**Proposition 2.7:**
Let U be a maximal m-open set and x an element of U. Then, U = ∪{W | W is an m-open neighborhood of x such that W∪U ≠ X}. Since U is maximal open then ∪{W | W is an m-open neighborhood of x such that W∪U ≠ X} ⊆ U. Therefore, we have the result. ■

3. m-Closure, m-Interior, and Maximal m-Open Sets

In this section we computation the closure of maximal m-open sets and the m-closure, the m-interior of other sets.

**Theorem 3.1:**
Let U be a maximal m-open set and x be an element of X−U. Then, X−U⊆W for any m-open neighborhood W of x.

**Proof:**
Suppose that X−U≠W, for some m-open neighborhood W of x. Then W∩U≠X which contradicts that U is maximal m-open. Therefore X−U⊆W. ■

**Corollary 3.2:**
Let U be a maximal m-open set. Then, following (i) or (ii) of the following holds:
(i) For each x∈X−U and each m-open neighborhood W of x, W=X;
(ii) There exists an m-open set W such that X−U⊆W and W⊆X.

**Proof:**
If (i) does not hold, then there exists an element x of X−U and an m-neighborhood W of x such that W⊆X. By Theorem 3.1, we have X−U⊆W. ■

**Corollary 3.3:**
Let U be a maximal m-open set. Then, following (i) or (ii) of the following holds:
(i) For each x∈X−U and each m-open neighborhood W of x, we have X−U⊆W.
(ii) There exists an m-open set W such that X−U=W ≠ X.

**Proof:**
Assume that (ii) does not hold. Assume that x∈X−U, then, by Theorem 3.1, we have X−U⊆W each m-open neighborhood W of x. ■

**Theorem 3.4:**
Let U be a maximal m-open set. Then, m-Cl(U) = X or m-Cl(U) = U.

**Proof:**
Since U is a maximal m-open set, then either one of the following cases (i) and (ii) occur by Corollary 3.3: (i) for each x∈X−U and each m-open neighborhood W of x, we have X−U⊆W. In this case let x be any element of X−U and W any m-open neighborhood of x. Since X−U≠W, we have W∩U≠∅ for any m-open neighborhood W of x. Hence, X−U⊆m-Cl(U). Since X=U∪(X−U) ⊆ U∪m-Cl(U) = m-Cl(U) ⊆ X, we have m-Cl(U)=X; (ii) If there exists an m-open set W such that X−U=W ≠ X. X−U=W is an m-open set, then U is an m-closed set. Therefore, U=m-Cl(U). ■

**Theorem 3.5:**
Let U be a maximal m-open set. Then m-Int(X−U) = X−U or m-Int(X−U) = ∅.

**Proof:**
By Corollary 3.3, we have either (i) m-Int(X−U) = ∅ or (ii) m-Int(X−U) = X−U. ■

**Theorem 3.6:**
Let U be a maximal m-open set and S a nonempty subset of X−U. Then m-Cl(S) = X−U.
**Definition 3.11:**
A subset $M$ of a space $(X, m_X)$ is called an $m$-preopen set if $M \subseteq m\text{-Int}(m\text{-Cl}(M))$.

**Theorem 3.12:**
Let $U$ be a maximal $m$-open set and $M$ any subset of $X$ with $U \subseteq M$. Then, $M$ is a $m$-preopen set.

**Proof:**
If $M = U$, then $M$ is an $m$-open set. Therefore, $M$ is an $m$-preopen set. Otherwise, $U \not\subseteq M$, then $m\text{-Int}(m\text{-Cl}(M)) = m\text{-Int}(X) = X \supseteq M$ by Corollary 3.7. Therefore, $M$ is an $m$-preopen set.

**Corollary 3.13:**
Let $U$ be a maximal $m$-open set. Then, $X \setminus \{a\}$ is an $m$-preopen set for any element $a$ of $X \setminus U$.

**Proof:**
Since $U \subseteq X \setminus \{a\}$ by our assumption, we have the result by Theorem 3.12.

**4. Fundamental Properties of Radicals**
In this section, we introduce the concept of radicals of maximal $m$-open sets and some of its properties.

**Definition 4.1:**
Let $U_\lambda$ be a maximal $m$-open set for any element $\lambda$ of $\Lambda$. Let $\mu = \{U_\lambda \mid \lambda \in \Lambda\}$, $\cap \mu = \cap_{\lambda \in \Lambda} \{U_\lambda\}$, is called the radical of $\mu$. The intersection of all maximal ideals of a ring $\mathcal{R}$ is called the (Jacobson) radical of $\mu$ [3].

Following this terminology in the theory of rings, we use the terminology “radical” for the intersection of maximal $m$-open sets. The symbol $\Lambda \setminus \Gamma$ means difference of index sets; namely, $\Lambda \setminus \Gamma = \Lambda - \Gamma$, and the cardinality of a set $\Lambda$ is denoted by $|\Lambda|$ in the following arguments.

**Theorem 4.2:**
Assume that $|\Lambda| \geq 2$. Let $U_\lambda$ be a maximal $m$-open set for any element $\lambda$ of $\Lambda$ and $U_\lambda \neq U_\mu$ for any elements $\lambda$ and $\mu$ of $\Lambda$ with $\lambda \neq \mu$. (i) Let $\mu$ be any element of $\Lambda$. Then, $X \setminus \cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \subseteq U_\mu$. (ii) Let $\mu$ be any element of $\Lambda$. Then, $\cap_{\lambda \in \Lambda \setminus \{\mu\}} U_\lambda \neq \emptyset$. 

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**Proof:**
Since $\emptyset \neq S \subseteq X - U$, we have $W \cap S \neq \emptyset$ for any element $x$ of $X - U$ and any $m$-open neighborhood $W$ of $x$ by Theorem 3.1. Then $X - U \in m\text{-Cl}(S)$. Since $X - U$ is an $m$-closed set and $S \subseteq X - U$, then $m\text{-Cl}(S) \subseteq m\text{-Cl}(X - U) = X - U$.

**Corollary 3.7:**
Let $U$ be a maximal $m$-open set and $M$ a subset of $X$ with $U \not\subseteq M$. Then, $m\text{-Cl}(M) = X$.

**Proof:**
Since $U \not\subseteq M \subseteq X$, there exists a nonempty subset $S$ of $X - U$ such that $M = U \cup S$. Hence, we have $m\text{-Cl}(M) = m\text{-Cl}(S \cup U) = m\text{-Cl}(S) \cap m\text{-Cl}(U) \supseteq (X - U) \cup U = X$ by Theorem 3.6. Therefore, $m\text{-Cl}(M) = X$.

**Corollary 3.8:**
Let $U$ be a maximal $m$-open set and assume that the subset $X - U$ has at least two elements. Then, $m\text{-Cl}(X \setminus \{a\}) = X$ for any element $a$ of $X - U$.

**Proof:**
Since $U \not\subseteq X \setminus \{a\}$ by our assumption, we have the result by Corollary 3.7.

**Theorem 3.9:**
Let $U$ be a maximal $m$-open set and $N$ a proper subset of $X$ with $U \subseteq N$. Then $m\text{-Int}(N) = U$.

**Proof:**
If $N = U$, then $m\text{-Int}(N) = m\text{-Int}(U) = U$.
Otherwise $N \neq U$, and hence $U \not\subseteq N$. It follows that $U \subset m\text{-Int}(N)$. Since $U$ is a maximal $m$-open set, we also $m\text{-Int}(N) \subseteq U$. Therefore, $m\text{-Int}(N) = U$.

**Theorem 3.10:**
Let $U$ be a maximal $m$-open set and $S$ a nonempty subset of $X - U$. Then $X - m\text{-Cl}(S) = m\text{-Int}(X - S) = U$.

**Proof:**
Since $U \subseteq X - S \subseteq X$ by our assumption, we have the result by Theorems 3.6 and Theorem 3.9.
Proof:
Let μ be any element of Λ. (i) Since $X-U_μ \subseteq U_μ$ for any element λ of Λ with $\lambda \neq μ$. Then, $X-U_μ \subseteq \cap_{\lambda \in Λ\mid μ}U_λ$. Therefore, we have $X-\cap_{\lambda \in Λ\mid μ}U_λ \subseteq U_μ$. (ii) If $\cap_{\lambda \in Λ\mid μ}U_λ = \emptyset$, we have $X = U_μ$ by (i). This contradicts our assumption that $U_μ$ is maximal m-open set. Therefore, we have $\cap_{\lambda \in Λ\mid μ}U_λ \neq \emptyset$.

Corollary 4.3:
Let $U_λ$ be a maximal m-open set for any element λ of Λ and $U_λ \neq U_μ$ for any elements λ and μ of Λ with $\lambda \neq μ$. If $|Λ| \geq 3$, then $U_λ \cap U_μ \neq \emptyset$ for any elements λ and μ of Λ with $\lambda \neq μ$.

Proof:
By Theorem 4.2(ii), we have the result.

Corollary 4.4:
(a decomposition theorem for maximal m-open set). Assume that $|Λ| \geq 2$. Let $U_λ$ be a maximal m-open set for any element λ of Λ and $U_λ \neq U_μ$ for any elements λ and μ of Λ with $\lambda \neq μ$. Then, for any element μ of Λ, $U_μ = (\cap_{λ \in Λ\mid μ}U_λ) \cap (X-\cap_{λ \in Λ\mid μ}U_λ)$.

Proof:
Let μ be an element of Λ. By Theorem 4.2 (1), we have:
$$(\cap_{λ \in Λ\mid μ}U_λ) \cap (X-\cap_{λ \in Λ\mid μ}U_λ) = ((\cap_{λ \in Λ\mid μ}U_λ) \cap U_μ) \cup (X-\cap_{λ \in Λ\mid μ}U_λ) = \cap_{λ \in Λ\mid μ}U_λ \cup (X-\cap_{λ \in Λ\mid μ}U_λ) = \cap_{λ \in Λ\mid μ}U_λ$$. Therefore, we have $U_μ = (\cap_{λ \in Λ\mid μ}U_λ) \cup (X-\cap_{λ \in Λ\mid μ}U_λ)$.

Theorem 4.5:
Let $U_λ$ be a maximal open set for any element λ of Λ and $U_λ \neq U_μ$ for any elements λ and μ of Λ with $\lambda \neq μ$. Assume that $|Λ| \geq 2$. Let μ be any element of Λ. Then, $\cap_{λ \in Λ\mid μ}U_λ \neq U_μ$ for any elements λ and μ of Λ with $\lambda \neq μ$.

Proof:
Let μ be any element of Λ. If $\cap_{λ \in Λ\mid μ}U_λ \subseteq U_μ$, then we see that $X = (X-\cap_{λ \in Λ\mid μ}U_λ) \cup \cap_{λ \in Λ\mid μ}U_λ \subseteq U_μ$ by Theorem 4.2(i). This contradicts our assumption. If $U_μ \cap \cap_{λ \in Λ\mid μ}U_λ$, then we have $U_μ \subseteq U_λ$, and hence $U_μ = U_λ$ for any element λ of Λ\mid μ}. This contradicts our assumption that $U_μ \neq U_λ$ when $\lambda \neq μ$.

Corollary 4.6:
Let $U_λ$ be a maximal m-open set for any element λ of Λ and $U_λ \neq U_μ$ for any elements λ and μ of Λ with $\lambda \neq μ$. If Γ is a proper nonempty subset of Λ, then $\cap_{λ \in Λ\mid Γ}U_λ \neq \cap_{λ \in Λ\mid Γ}U_λ$.

Proof:
Let γ be any element of Γ. We see $\cap_{λ \in Λ\mid Γ}U_λ = \cap_{λ \in Λ\mid Γ\setminus γ}U_λ$ by Theorem 4.5. Therefore we see $\cap_{λ \in Λ\mid Γ}U_λ \neq \cap_{γ \in Γ\mid Γ\setminus γ}U_γ$. On the other hand, since $\cap_{γ \in Γ\mid Γ\setminus γ}U_γ \neq \cap_{λ \in Λ\mid Γ}U_λ$, we have $\cap_{γ \in Γ\mid Γ\setminus γ}U_γ \neq \cap_{λ \in Λ\mid Γ}U_λ$.

Theorem 4.7:
Let $U_λ$ be a maximal m-open set for any element λ of Λ and $U_λ \neq U_μ$ for any elements λ and μ of Λ with $\lambda \neq μ$. If Γ is a proper nonempty subset of Λ, then $\cap_{λ \in Λ\mid Γ}U_λ \neq \cap_{γ \in Γ\mid Γ\setminus γ}U_γ$.

Proof:
By Corollary 4.6, we have $\cap_{λ \in Λ\mid Γ}U_λ \neq \cap_{γ \in Γ\mid Γ\setminus γ}U_γ$.

Theorem 4.8:
Assume that $|Λ| \geq 2$. Let $U_λ$ be a maximal m-open set for any element λ of Λ and $U_λ \neq U_μ$ for any elements λ and μ of Λ with $\lambda \neq μ$. If $\cap_{λ \in Λ\mid μ}U_λ = \emptyset$, then $\{U_λ | λ \in Λ\}$ is the set of all maximal m-open sets of X.

Proof:
If there exists another maximal m-open set $U_ν$ of X, which is not equal to $U_λ$ for any element λ of Λ, then $\emptyset = \cap_{λ \in Λ\mid μ}U_λ = \cap_{λ \in Λ\mid Γ\setminus γ}U_λ$. By Theorem 4.2(ii), we see $\cap_{λ \in Λ\mid Γ\setminus γ}U_λ \neq \emptyset$. This contradicts our assumption.

Proposition 4.9:
Let $U_λ$ be a set for any element λ of Λ. If $m-Cl(\cap_{λ \in Λ\mid μ}U_λ) = X$, then $m-Cl(U_λ) = X$ for any element λ of Λ.

Proof:
We see that $X = m-Cl(\cap_{λ \in Λ\mid μ}U_λ) \subseteq m-Cl(U_λ)$. It follows that $m-Cl(U_λ) = X$ for any element λ of Λ.

References


الخلاصة
في هذا العمل نحن نقدم مفهوم المجموعة المفتوحة العظمى في فضاءات البنية الصغرى و ندرس بعض الخصائص الأساسية لهذه المجموعة في تلك الفضاءات.