Abstract

Bifurcation of critical point of three dimensional differential equation autonomous system

\[ F(x) = \frac{dx}{dt} \]

with \( F(x) = (F_1, F_2, F_3) \) and \( X = (x_1, x_2, x_3) \) with the square term \( (x_1^2) \) was studied by Paul Phillips on and Peter Schuster. In this paper bifurcation of three-dimensional autonomous system with cubic term \( (x_3^3) \) and the two parameters is studied. The studying is carried out by transforming to non-autonomous system and using the averaging method.

Introduction

Non-linear dynamics has some claim to be the most ancient of scientific problems and in the study of such anomalous phenomena, one problem which should be studied is bifurcation and periodic motions near an equilibrium solution which can be evolve. The bifurcation theory as one of the main contents of non-linear science, has developed rapidly in the past three decades. Oscillations described by autonomous three dimensional differential equations and chaos at critical parameter values. Paul Phillipson and Peter Schuster studied autonomous three-dimensional differential equation systems of the form

\[ \frac{dx}{dt} = f(x) \quad \text{where} \quad f(x) = (f_1(x), f_2(x), f_3(x)) \]

and \( X = (x_1, x_2, x_3) \)

with the non-linear term \( (x_1^2) \) which can display a rich diversity of periodic multiple periodic and chaotic flows dependent upon the specific values of one or more control parameter. Analysis of the equations linearized around a fixed point serve to provide local solutions which are stable or unstable depending upon whether all eigenvalues are either negative or have negative real parts. In this paper we study bifurcation of autonomous three-dimensional differential equation with the nonlinear term \( (x_3^3) \) and two control parameters \( (r) \) and \( (s) \). We get an important relation between these parameters which can determine the bifurcation of this system.

The Dynamical system

We consider the following three-dimensional nonlinear autonomous system of differential equations with two parameters

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= z \\
\frac{dz}{dt} &= -(z + sy - rx + x^3)
\end{align*}
\]

(1)

Jack K. Hale and Hussein. K. in [3] discuss the stability and bifurcation of nonlinear autonomous system by using the following important theorem:

Theorem 1.1

Let \( X^1 = F(x) \) be nonlinear autonomous system and \( F \) be \( C^1 \) function. If all the eigenvalues of the Jacobian matrix \( (DF) \) at the critical point \( X_0 \) have negative real part then \( (x_0) \) is asymptotically stable.

System (1) has three critical points \((0,0,0)\), \((\sqrt{r},0,0)\), \((\sqrt{r},0,0)\). The first point is unstable and at the second point \((\sqrt{r},0,0)\), the system have one negative eigenvalue \( \frac{1}{6}(36s - 216r) \) and two complex conjugate eigenvalues with positive real part \((-\frac{1}{12}(36s - 216r) + \sqrt{\frac{5}{2}}I)\) where \( r, s \). To discuss the bifurcation of this system we follow the averaging procedure which is used for treating Duffing and modified Duffing equation [4] but there is two differences:
First: system (1) is autonomous system that is there is no external force, so we must assume the forcing function corresponding to the function of the period-one orbit as a driving force

Second: the averaging method assume that the external force constitutes a perturbation on linear oscillator the condition of smallness can be imposed on external force and here the force is self – generated and determined by the magnitude of control parameters which may be small or large [2].

When \( r < s \) the real eigenvalue is positive, so we can not reached to stability state of this system in this case.

The Forcing Function

First we assume that the solution of the autonomous system can be split in to two parts \( x = x_1 + x_2 \), \( y = y_1 + y_2 \), \( z = z_1 + z_2 \) and \( (x_1, y_1, z_1) \) satisfies exactly the autonomous system

\[
\begin{align*}
\frac{dx_1}{dt} &= y_1 \\
\frac{dy_1}{dt} &= z_1 \\
\frac{dz_1}{dt} &= -[z_1 + sy_1 - rx_1 + x^3] 
\end{align*}
\]

For which the system solution is a period one-orbit, and since the system is non linear, the equation for \((x_2, y_2, z_2)\) is non autonomous by the solution \( x_1 \)

\[
\begin{align*}
\frac{dx_2}{dt} &= y_2 \\
\frac{dy_2}{dt} &= z_2. \\
\frac{dz_2}{dt} &= -[z_2 + sy_2 - [r - x_1(t) - 3x_1(t)x_2(t)]x_2 + x_2] 
\end{align*}
\]

So the period one solution provides the forcing term for driving the system into dynamical states other than period-one. We shall assume the forcing function in the form

\[ X_1(t) = [x_0 + \beta \cos(wt)] + f(t) \] \( (4a) \)

Where \( x_0, \beta \) and \( w \) are constants to be determined and \( f(t) \) represents the difference between the exact solution and the approximate solution in brackets.

Substitution in to equation (2) and after some steps we get

\[
\begin{align*}
[((w^3 - sw)B \sin(wt) + [-rB - Bw^2 + 3Bx_0^2] \cos(wt) + [x_0^3 - r x_0 - 3B^2 x_0] + \frac{\frac{d^3f}{dt^3} + \frac{d^2f}{dt^2} + \frac{df}{dt}}{2})
\end{align*}
\]

and here the large bracket vanish results in the unique determination of the constants in the terms of the control parameters [2]:

\[
\begin{align*}
w^3 - sw = 0 & \quad \text{then } w = \sqrt{s} \quad w = 0 \\
r - w^2 - x_0^2 = 0 & \quad \text{then } x_0 = \sqrt{\frac{r + s}{3}}
\end{align*}
\]

So

\[
\begin{align*}
\beta^2 &= \frac{2(r - x_0^2)}{3} \\
X_0 &= \sqrt{\frac{4r - 2s}{9}}
\end{align*}
\]

Substitution \( X_0 \) in \( \beta^2 \) then \( \beta = \sqrt{\frac{4r - 2s}{9}} \)

The Averaging procedure

The next step is to cast the equations for the period – two orbit equation (3) into a form suitable for the averaging procedure. Substitution equation (4b) for the period - one orbit into equation (3) gives:

\[
\begin{align*}
\frac{dx_2}{dt} &= y_2, \quad \frac{dy_2}{dt} = z_2 \quad (5a) \\
\frac{dz_2}{dt} &= \left\{ \frac{z_2 + sy_2 - [r - (\delta_1 \cos(wt))]X_2 + X_2^3}{(\delta_1 \cos(wt) + 3X_2)} \right\} \\
\end{align*}
\]

where

\[
\begin{align*}
\delta &= \sqrt{\frac{r - s}{3}}, \quad \delta_1 = \sqrt{\frac{4r - 2s}{9}} \quad (5b)
\end{align*}
\]

which is a non autonomous equation T–periodic in \( t \) of the same formal structure as that of the forced Duffing and modified
Duffing’s equation [4]. We introduce the Vanderpal transformation from the coordinates \( X_2, Y_2 \) to new coordinates \( u, v \) parameterized by \( k \) [4].

\[
X_2 = u \cos(kwt) - v \sin(kwt) \quad \quad (6)
\]

\[
y_2 = -kwu \sin(kwt) - kwv \cos(kwt)
\]

Thus

\[
Z_2 = \frac{dy_2}{dt} = [(kw)^2u + kw \frac{du}{dt}] \cos(kwt) - [kw^2v - kw \frac{dv}{dt}] \sin(kwt)
\]

\[
\begin{pmatrix}
X_2 \\
y_2
\end{pmatrix} = \begin{pmatrix}
\cos(kwt) & -\sin(kwt) \\
-kw \sin(kwt) & -kw \cos(kwt)
\end{pmatrix} \begin{pmatrix}
u \\
v
\end{pmatrix}
\]

.............................................................................. (7)

The inverse transformation of equation (7) is

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}
\cos(kwt) & -\frac{\sin(kwt)}{kw} \\
-kw \sin(kwt) & -\frac{\cos(kwt)}{kw}
\end{pmatrix} \begin{pmatrix} X_2 \\
y_2
\end{pmatrix}
\]

................................................. (8)

Differentiation of the inverse transformation equation (8) for t gives

\[
\begin{pmatrix}
du \\
dv
\end{pmatrix} = \begin{pmatrix}
-kwx_2 + \frac{z_2}{kw} \frac{\sin(kwt)}{\cos(kwt)} \\
-kw \frac{du}{dt} + \frac{1}{kw} \frac{dz_2}{dt} \sin(kwt)
\end{pmatrix}
\]

................................................. (9)

Substitution equation (6) into equation (9) and differentiation for t gives

\[
\begin{array}{c}
\frac{d^2u}{dt^2} = kw \frac{dv}{dt} - [kwy_2 + \frac{1}{kw} \frac{dz_2}{dt}] \sin(kwt) \quad (10)
\\
\frac{d^2v}{dt^2} = -kw \frac{du}{dt} + [kwy_2 + \frac{1}{kw} \frac{dz_2}{dt}] \cos(kwt)
\end{array}
\]

Thus from equation (10) and (11) we get the following system

\[
\begin{pmatrix}
\frac{d^2u}{dt^2} \\
\frac{d^2v}{dt^2}
\end{pmatrix} = \begin{pmatrix}
-kwx_2 + \frac{z_2}{kw} \frac{\sin(kwt)}{\cos(kwt)} \\
-kw \frac{du}{dt} + \frac{1}{kw} \frac{dz_2}{dt} \sin(kwt)
\end{pmatrix}
\]

................................. (12)

By substitution equations (5a), (6) in equation (12) and setting \( k = \frac{1}{2} \) as a first step and defining \( cn = \cos \left( \frac{nwt}{2} \right) \), \( Sn = \sin \left( \frac{nwt}{2} \right) \), \( (n=1,2,\ldots) \) as a second step and employing some trigonometric relations as a third step we can write system (12) as

\[
\begin{pmatrix}
\frac{d^2u}{dt^2} \\
\frac{d^2v}{dt^2}
\end{pmatrix} = \begin{pmatrix}
A \frac{du}{dt} + \beta u + \frac{1}{2} u^2 + D \frac{u^3}{v^3} \\
B \frac{dv}{dt} + \gamma v + \frac{1}{2} v^2 + E \frac{v^3}{u^3}
\end{pmatrix}
\]

................................................. (13)

The coefficients may be written in matrix form by sorting out the terms involving periodic functions

\[
A = A_1 + A_2 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} c_2 & -\frac{1}{2} s_2 \\
\frac{1}{2} & -\frac{1}{2} c_2 & \frac{1}{2} s_2 \\
\frac{1}{2} & -\frac{1}{2} s_2 & -\frac{1}{2} c_2
\end{pmatrix}
\]

\[
B = B_1 + B_2 = \begin{pmatrix}
\frac{3s}{8} & \frac{w}{4} \left( \frac{\delta + \delta_1}{2} \right)^2 + \frac{s^2 - r}{2w} & \frac{3s}{8}
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
\frac{3s c_1}{w} & \frac{3s_1 c_1}{w} & -\frac{3s c_1}{w} - \frac{3s_1 c_2}{w} + \frac{1}{2} c_4 \\
\frac{3s_1 c_1 + 3s c_2}{w} & \frac{3s c_1 + 3s_1 c_2}{w} & \frac{1}{2} c_4 \\
\frac{3s_1 c_1 + 3s c_2}{w} & \frac{9s_1 c_1 c_2 + 3s_1}{w} & \frac{9s_1 c_1 c_2}{w} - \frac{3s_1 c_2}{w}
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
\frac{s_1}{4w} & \frac{s_2}{2w} & \frac{3c_1}{4w} & \frac{3c_2}{4w} & \frac{s_3}{4w} & \frac{s_4}{4w} \\
\frac{c_1}{2w} & \frac{c_2}{2w} & -\frac{3s_1}{4w} & \frac{3s_1}{4w} & -\frac{3c_1}{2w} & -\frac{s_2}{2w} + \frac{s_4}{4w}
\end{pmatrix}
\]

we note that since equation (5a) is three dimensional, this equation under the Vanderpal transformation to equation (13) is necessarily of second order in the coordinates \( u, v \).

The solution of equations (13) and (14) will be affected by the application of the averaging procedure. This solution will be approximated by the solution of the averaged differential equation defined by
\[ \frac{d^2 z}{dt^2} = \frac{1}{T} \int_{t}^{T} F \left( \frac{dz}{dt}, z, t \right) dt \quad \text{(15a)} \]

where \( Z \in \mathbb{R}^2 \) and \( F \) is \( T \) periodic in \( t \) [1]. In our problem \( Z = (u, v) \) with \( T = 4 \pi /w \) upon carrying out this averaging process the coefficients of the quadratic and cubic terms \( c_{ij}, d_{ij} \) average to zero, so that the only terms that survive for the linear terms are the constant quantities of the matrix \( A_1 \)

\[ \frac{d^2 (u, v)}{dt^2} = \left( A_1 \frac{du}{dt} + B_1 \right) \left( \begin{array}{c} u \\ v \end{array} \right) \]

\[ \text{........................................ (15b)} \]

The next stage is to introduce the following non linear transformation from \( (u, v) \) to the coordinates \( (x, y) \) by generating functions \( u_1, u_2 \) which must be determined [1].

\[ u = x + u_1 \left( x, y, \frac{dx}{dt}, \frac{dy}{dt}, t \right) \]

\[ v = y + u_2 \left( x, y, \frac{dx}{dt}, \frac{dy}{dt}, t \right) \]

\[ \text{........................................ (18a)} \]

Then:

\[ \frac{du}{dt} = \frac{dx}{dt} + \frac{\partial u_1}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u_1}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial t} \cdot \frac{dt}{dt} + \]

\[ \frac{d\left( \frac{dx}{dt} \right)}{dt} \]

\[ \frac{dv}{dt} = \frac{dy}{dt} + \frac{\partial u_2}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u_2}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial t} \cdot \frac{dt}{dt} + \]

\[ \frac{d\left( \frac{dy}{dt} \right)}{dt} \]

\[ \text{........................................ (18b)} \]

and:

\[ \frac{d^2 u}{dt^2} = \frac{d^2 x}{dt^2} + \frac{\partial^2 u_1}{\partial t^2} \]

\[ \text{........................................ (18c)} \]

The terms in the partial derivative involving \( \left( \frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2} \right) \) as well as cross derivatives but they will make no contribution in the second averaging process [1]

Substitution (18b) and (18c) into equations 13 and 14 and grouping significant terms as follows:

\[ \frac{d^2 (x, y)}{dt^2} = A \frac{d^2 (u, v)}{dt^2} + B \frac{du}{dt} + C \frac{u^2}{v^2} + D \frac{u^3}{v^3} + \]

\[ \text{........................................ (18d)} \]

\[ A = A_1 + A_2, \quad B = B_1 + B_2 \]

5-**Averaging to the second order**

The construction of the appropriate stable period two solutions requires the extension of the averaging procedure which properly includes the rule of nonlinear terms and the corrected averaged linear terms.
But 
\[ \frac{d}{dt}(u) = \frac{d}{dt}(x) + W \frac{d}{dt}(y) + \frac{\partial}{\partial t} \Phi \left( u_1 \right) \]

Where 
\[ W = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix} \]

Being the Jacobean, so
\[ \frac{d^2}{dt^2} \begin{pmatrix} x \\ y \end{pmatrix} = A_1 \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} + A_2 \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} + A_3 \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + A_4 \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \]

\[ + A_5 \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + B_1 \begin{pmatrix} x \\ u_1 \end{pmatrix} + B_2 \begin{pmatrix} x \\ u_2 \end{pmatrix} \]

\[ + C \begin{pmatrix} x^2 \\ u_1^2 \\ u_2^2 \end{pmatrix} + D \begin{pmatrix} xu_1 \\ xu_2 \\ u_1 u_2 \end{pmatrix} \]

\[ + \begin{pmatrix} 3x^2 u_1 \\ 3x^2 u_2 \\ 6xu_1 u_2 \end{pmatrix} + \begin{pmatrix} 3xu_1 \\ 3xu_2 \\ 6xu_1 u_2 \end{pmatrix} \]

\[ \Rightarrow \frac{d^2}{dt^2} \begin{pmatrix} x \\ y \end{pmatrix} = (A_1 + A_2 W) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} \]

and the terms quadratic and cubic in \( u_1, u_2 \) have been neglected. This negligible implies equation (18a) represents a smooth near identity transformation in which the effect of these nonlinear functions, which are proportional to \( 2/(s+1) \), \( 1/s \) according to equations(20)and(21) are preservative. There are other terms in the expression (19a) which, when set equal to zero, provided the conditions for the determination of these generators according to
\[ \frac{\partial^2}{\partial t^2} \Phi \left( u_1 \right) = A_1 \frac{\partial}{\partial t} \Phi \left( u_1 \right) + \begin{pmatrix} f \\ g \end{pmatrix} \]

with 
\[ \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} (A_5 + B_2) \frac{d}{dt}(x) + C \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} + D \begin{pmatrix} 3x^2 y \\ 3xy^2 \\ y^3 \end{pmatrix} \end{pmatrix} \]

This is a linearly coupled system characterized by a pair of eigenvalues with negative real part. The asymptotic solutions for \( \frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t} \) are simply integrals over \( \tilde{f}, \tilde{g} \). Integrating \( \frac{\partial}{\partial t} \Phi \left( u_1 \right) \) with respect to 
\[ (t) \]

\[ u_1 = \frac{2}{1+s} \begin{pmatrix} M_1 \sin(wt) + M_2 \cos(wt) \\ + M_3 \sin(wt) + M_4 \cos(wt) \end{pmatrix} \]

\[ + M_5 \cos(2wt) + M_6 \sin(2wt) \]

\[ u_2 = \frac{2}{1+s} \begin{pmatrix} M_2 \sin(wt) + M_3 \cos(wt) \\ - M_1 \sin(wt) - M_2 \cos(wt) \end{pmatrix} \]

\[ - M_5 \cos(2wt) - M_6 \sin(2wt) \]

\[ \text{.................}(21) \]

Where
\[ M_1 = \frac{1}{2w} \frac{dx}{dt} \frac{1}{2w} \frac{dy}{dt} \frac{x^3 + y^3}{w^2} + \frac{3w}{2w} \frac{y}{s} - \frac{y}{4} \]

\[ + \frac{\delta^2}{4} + \frac{\delta^2}{2} - 2 \frac{w}{4} \frac{x + \delta^2}{2w} - \frac{x^3}{2w} + \frac{3s}{8} y \]

\[ M_2 = \frac{1}{2w} \frac{dx}{dt} \frac{1}{2w} \frac{dy}{dt} \frac{3\delta \delta}{4w} \frac{y^2}{x^3 - \frac{1}{2w}} \frac{2w}{y} \]

\[ - \frac{\delta^2}{2} + \frac{\delta^2}{2} - 2 \frac{w}{8} \frac{x + \frac{3w}{8} \frac{y}{8} - \frac{x - \frac{3w}{8} \frac{y}{4}}}{2s} \]

\[ \frac{M_3}{w} = \frac{6\delta}{w} x^2 \]

\[ \frac{M_4}{y^2} = \frac{-6\delta - 6\delta \delta}{y^2} \]

\[ \frac{M_5}{w} = \frac{-x^3}{8w} + \frac{y^3}{8w} - \frac{\delta \delta}{2w} x + \frac{\delta \delta}{2w} y \]

\[ \frac{M_6}{2w} \left[ \frac{1}{4} x + y \right] \left[ \frac{\delta^2}{4w} - \frac{\delta \delta}{2w} \right] \]

\[ + \frac{\delta \delta}{2w} + \frac{\delta \delta}{4w} + \frac{\delta \delta}{4w} y \]

\[ \text{.................}(22) \]

Now, for averaging procedure of equation (15a) subsisting generating function solutions equations (20) and (21) into equation 19 and
\[
\frac{d^2}{dt^2}(x) = L \frac{dx}{dt} + (M + N) \left( x, y \right) = \frac{\delta_1^2}{16s(s+1)} \frac{y}{x^2 + y^2} \left( x, y \right)
\]

where the matrices L, M and N are given by equation (23).

Now we want to find critical points \((x_0, y_0)\) of the averaged equation (24) by setting the derivatives equal to zero, defining \(x_0=r_0 \sin \theta_0\) and \(y_0=r_0 \cos \theta_0\) as follows:

\[
\begin{align*}
\frac{d^2}{dt^2}(x) &= 0 \\
\frac{dx}{dt} &= 0
\end{align*}
\]

Divided the last relation by \(r_0\) and setting the determinant of \(\sin \theta_0\) and \(\cos \theta_0\) equal to zero to get the relation between the parameters \(r, s, \) and \(r_0\) by obtained the quadratic form for \(\delta^2\) as follows:

\[
\delta^2 = \frac{(8 + 8s) \delta_1^4 + 576 \delta_1^3 \delta_{10}^2 + 144 \delta_1 \delta_{10}^2 + 295 s \delta_1^2 \delta_{10}}{128(s-1)^4 \delta_{10}^4 + s}
\]

The range of stability of the period two orbit is that the parameter range for which the solution is stable as determined by the eigenvalue structure of equation (24) linearized around the critical points [2]. Setting \(X = x_0 + x', \ y = y_0 + y\) the linear zed equations can be obtained as follows:

\[
\begin{align*}
\frac{dx'}{dt} &= \lambda_{11} x' + \lambda_{12} y' \\
\frac{dy'}{dt} &= m_{11} x' + m_{11} y' - \frac{\delta_1^2 (x_0 y_0 + y_0 x_0)}{16s \cos(s+1)}
\end{align*}
\]
We must find the solutions of the eigenvalue equation after transforming of system (26) to the system of first order equations, then the eigenvalue equation is:

\[ \lambda^4 + r_1 \lambda^3 + r_2 \lambda^2 + r_3 \lambda + r_4 = 0 \]

.................................(27)

where the coefficients are:

\[ r_1 = -\lambda_{11} - \lambda_{22} \]
\[ r_2 = \lambda_{11} \lambda_{22} - \lambda_{12} \lambda_{21} - m_{22} + n_{22} \]
\[ + \frac{\delta^2 (x_0 + 3 y_0^2)}{16 \text{os}(s + 1)} - m_{11} - n_{11} \]
\[ + \frac{\delta^2 (3 x_0 + y_0^2)}{16 \text{os}(s + 1)} \]
\[ r_3 = -\lambda_{21} m_{12} - \lambda_{21} n_{12} + \lambda_{21} \]
\[ - \lambda_{11} m_{22} - \lambda_{11} n_{22} - \lambda_{11} \]
\[ - \lambda_{12} m_{21} - \lambda_{12} n_{21} + \lambda_{12} \]
\[ - m_1 \lambda_{22} + n_1 \lambda_{22} - \lambda_{22} \]
\[ r_4 = (m_{11} n_{11} + \delta^2 (x_0 + y_0^2))(m_{22} + n_{22}) \]
\[ - \frac{\delta^2 (x_0 + 3 y_0^2)}{16 \text{os}(s + 1)} + (m_{12} + n_{12}) \]
\[ - \frac{\delta^2 (x_0 + y_0^2)}{8 \text{os}(s + 1)}(m_{21} + n_{21}) - \frac{\delta^2 (x_0 + y_0^2)}{8 \text{os}(s + 1)} \]

.................................(28)

The definition of the Vanderpol transformation to coordinates \( u, v \) of equation (6) combined with the subsequent transformation to the coordinates \( x, y \) according to the equation (18) implies that vicinal to the onset of periods \( 1 \rightarrow 2 \) bifurcation the contribution of the period two orbit of equation (7) is given approximately by

\[ x(t) = \left( \frac{1}{2} r+s \right) + \left( \frac{1}{9} r-2s \right) \cos(\omega t) \]
\[ + r_0 \sin(\theta) \]

.................................(29)

The range of stability of the period two orbits is that the range of the parameter \( r_0 \) for which all the solutions of equation (27) negative or have negative real parts. That is \( r_k \geq 0 \), \( k=1,2,3,4 \). At \( r_0 = 0 \), we can obtain two solutions of \( \delta^2 \)

\[ \delta^2 = \frac{(8+8s)\delta_1^2 + 8(3s^2+2s+17)\delta^2_1 + s^5}{128(s-1)\delta_1^4} \]

By using the relation (5b) there are two values of \( r \)

\[ r = \sqrt[4]{65536+65536} S^2 - 131072 S \]

.................................(30)

And the coefficients \( r_1, r_2, r_3, r_4 \) of the eigenvalue equation are;

\[ 65536+65536 \]
\[ r_1 = 1 - \frac{1-\sqrt{s+3s}s}{8(\sqrt{s+s}s)} + \frac{8s-8s}{18(s+s^2)} \]
\[ r_2 = \left( \frac{1}{2} + \frac{1-\sqrt{s+3s}s}{\sqrt{8(8+8s)s}} \right) - \frac{8s-8s}{18(s+s^2)} \frac{(4r-25)(r-s)}{27s(s+1)} \]
\[ r_3 = \frac{1}{2\sqrt{8s}(1+s)} - \frac{8s-8s}{27s\sqrt{s}(s+1)}(2r-s)(r-s) \]
\[ r_4 = \frac{1}{2916s^3}(s+1)^2 \frac{(4r-2s)^2(r-2)^2}{2}\frac{(2s+1)}{10} \]

We use maple 9 program to find the solutions of the eigenvalue equation (27), we find that as \( s<0.543=M_0 \), equation (27) have two solutions with positive real part and when \( s \) increase up to \( s=0.543 \) we get negative real part for all solutions of this equation. This value of \( s \) defines the point at which the points of bifurcation starts and less than which the period- one orbit anatomists stability. As increases the two pairs of solutions have negative real parts. So that the period -two orbits are stable:
Conclusions
The basic idea for studying bifurcation of three-dimensional autonomous system of differential equations with two parameters is to transforming it to a non autonomous system and then by important relation between the two parameters which determined the bifurcation state. In this paper bifurcation of system is studying. We need to consider averaging to the first and second order to get the approximate solution (29) the stability period –two orbits accurse if the value of the control parameter s increase up s0 =0.543, this value is less than the value of the some parameter in system with square term, so we can conjecture that the value of the control parameter s is connected with the power of x in the third equation of system (1). The order of the averaging method is not related with power of the variable x.

References