Pairwise Separation Axioms And Compact Double Topological Spaces

By
Rewayda Razaq Mohsin & Enas Y. Abdalla
Ahmed M. Rajab
University of Kufa, Najaf, Iraq

Abstract

The concept of intuitionistic (double) topological spaces was introduced by Çoker 1996. The aim of this paper is to give a nation of pairwise compactness for double topological spaces and some separation axioms .

الملخص :

فكرة فضاء التبولوجى المضاعف قدم من قبل Coker في العام 1996 . الهدف من هذا البحث هو تقديم تعريف التراص الزوجى للفضاءات التبولوجية المضاعفة وبعض بديهيات الفصل.

1-Introduction

The concept of a fuzzy topology was introduced by Change in 1968 [2] after the introduction of fuzzy sets by Zadeh in 1965. Later this concept was extended to intuitionistic fuzzy topological spaces by Çoker in [4]. In [5] Coker studied continuity, connectedness, compactness and separation axioms in intuitionistic fuzzy topological spaces. In this paper we follow the suggestion of J.G. Garcia and S.E. Rodabaugh [7] that (double fuzzy set)is a more appropriate name than (intuitionistic fuzzy set ) ,and therefore adopt the term (double-set) for the intuitionistic set , and (double-topology) for the intuitionistic topology of Dogan Çoker , (this issue) we denote by Dbl-Top the construct (concrete texture over Set ) whose objects are pairs \((X, \tau)\) where \(\tau\) is a double-topology on \(X\) .In Section three we discuss making use of this relation between bitopological spaces and double- topological spaces , we generalize a nation of compactness for double- topological space in section four with some theorems about \(T_1\), \(T_2\), \(T_3\).
2-Preliminaries

Throughout the paper by X we denote a non-empty set. In this section we shall present various fundamental definitions and propositions. The following definition is obviously inspired by Atanassov [1].

2.1. Definition. [8] A double-set (Ds in brief) A is an object having the form $A=\langle x,A_1,A_2 \rangle$. Where $A_1$ and $A_2$ are subsets of $X$ satisfying $A_1 \cap A_2 = \emptyset$. The set $A_1$ is called the set of members of $A$, while $A_2$ is called the set of non-members of $A$.

Throughout the remainder of this paper we use the simpler $A=(A_1,A_2)$ for a double-set.

2.2. Remark. Every subset $A$ of $X$ may obviously be regarded as a double-set having the form $A'=(A,A^c)$, where $A^c=X\setminus A$ is the complement of $A$ in $X$.

We recall several relations and operations between DS’s as follows:

2.3. Definition. [8] Let the DS’s $A$ and $B$ on $X$ be the form $A=(A_1,A_2)$, $B=(B_1,B_2)$, respectively. Furthermore, let $\{A_j : j \in J\}$ be an arbitrary family of DS’s in $X$, where $A_j=(A^{(i)}_j, A^{(2)}_j)$. Then

(a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
(c) $\overline{A} = (A_2, A_1)$ denotes the complement of $A$;
(d) $\bigcap A_j = (\bigcap A^{(i)}_j, \bigcup A^{(2)}_j)$;
(e) $\bigcup A_j = (\bigcup A^{(i)}_j, \bigcap A^{(2)}_j)$;
(f) $\bigcap A = (A_1, A^c)$;
(g) $\bigcap A = (A^c_2, A_2)$;
(h) $\phi = (\emptyset, X)$ and $\emptyset = (X, \emptyset)$. 

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In this paper we require the following:

(i) \( A = (A_1, \emptyset) \), and
(ii) \( A = (\emptyset, A_2) \).

Is call the image and preimage of DS’ s under a function

2.4. Definition. [3,8] Let \( x \in X \) be a fixed element in \( X \). Then:

(a) The DS given by \( x = (\{x\}, \{x\}^c) \) is called a double–point (DP in brief \( X \)).

(b) The DS \( x = (\emptyset, \{x\}^c) \) is called a vanishing double-point (VDP in brief \( X \)).

2.5. Definition. [3,8]

(a) Let \( x \) be a DP in \( X \) and \( A=(A_1, A_2) \) be a DS in \( X \). Then \( x \in A \iff x \in A_i \).

(b) Let \( x \) be a VDP in \( X \) and \( A=(A_1, A_2) \) a DS in \( X \). Then \( x \in A \iff x \notin A_2 \).

It is clear that \( x \in A \iff x \subseteq A \) and that \( x \notin A \iff x \subseteq A \).

2.6. Definition. [10] A double-topology (DT in brief) on a set \( X \) is a family \( \tau \) of DS’s in \( X \) satisfying the following axioms:

**T1:** \( \emptyset, X \in \tau \),

**T2:** \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \),

**T3:** \( \bigcup G_j \in \tau \) for any arbitrary family \( \{G_j : j \in J\} \subseteq \tau \).

In this case the pair \( (X, \tau) \) is called a double-topological space (DTS in brief), and any DS in \( \tau \) is known as a double open set (DOS in brief). The complement \( \overline{A} \) of a DOS \( A \) in a DTS is called a double closed set (DCS in brief) in \( X \).

2.7. Definition. [10] Let \( (X, \tau) \) be an DTS and \( A = (A_1, A_2) \) be a DS in \( X \).

Then the interior and closure of \( A \) are defined by:

\[
\text{int}(A) = \bigcup \{G : G \text{ is a DOS in } X \text{ and } G \subseteq A\},
\]

\[
\text{cl}(A) = \bigcap \{(H : H \text{ is a DCS in } X \text{ and } A \subseteq H)\},
\]

respectively.
It is clear that $\text{cl}(A)$ is a DCS in $X$ and $\text{int}(A)$ a DOS in $X$. Moreover $A$ is a DCS in $X$ iff $\text{cl}(A) = A$, and $A$ is a DOS in $X$ iff $\text{int}(A) = A$.

2.8. Example. [5] Any topological space $(X, \tau)$ gives rise to a DT of the form $\tau = \{A^c : A \in \tau\}$ by identifying a subset $A$ in $X$ with its counterpart $A^c = (A, A^c)$, as in Remark 2.2.

3. The Construction of Dbl-Top and Bitop:

We begin by recalling the following results which associates a bitopology with a double topology.

3.1. Proposition. [5] Let $(X, \tau)$ be a DTS.

(a) $\tau_1 = \{A_1 : \exists A_2 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ is a topology on $X$.

(b) $\tau_2^c = \{A_2 : \exists A_1 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ is the family of closed sets of the topology $\tau_2 = \{A_2^c : \exists A_1 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ is a topology on $X$.

(c) Using (a) and (b) we may conclude that $(X, \tau_1, \tau_2)$ is a bitopological space.

3.2. Proposition. Let $(X, u,v)$ be a bitopological space. Then the family

$$\{(U, V^c) : U \in u, V \in v, U \subseteq V\}$$

is a double topology on $X$.

Proof: The condition $U \subseteq V$ ensures that $U \cap V^c = \emptyset$, while the given family contains $\emptyset$ because $\emptyset \in u, v$, and it contains $X$ because $X \in u, v$. Finally this family is closed under finite intersections and arbitrary unions by Definition 2.3 (d,e) and the corresponding properties of the topologies $u$ and $v$.

3.3. Definition. Let $(X, u,v)$ be a bitopological space. Then we set

$$\tau_{uv} = \{(U, V^c) : U \in u, V \in v, U \subseteq V\}$$

and call this the double topology on $X$ associated with $(X, u,v)$.

3.4. Proposition. If $(X, u,v)$ is a bitopological space and $\tau_{uv}$ the corresponding DT on $X$, then

$$(\tau_{uv})_1 = u \text{ and } (\tau_{uv})_2 = v.$$
Proof. \( U \in u \) implies \((U, \phi) \in \tau_{uv}\) since \( U \subseteq X \in v, sou \subseteq (\tau_{uv})\). Conversely, take \( U \in (\tau_{uv})\). Then \((U, B) \in \tau_{uv}\) for some \( B \subseteq X\), and now \( U \in u\). Hence \((\tau_{uv})_u \subseteq u\), and the first equality is proved.

The proof of the second equality may be obtained in a similar manner, and we omit the details. Now we define double compact set and we use the link between bitopological space and double topological space to established some theorems.

4. Pairwise Double Compact Set:

4.1. Definition. By an double open cover of a subset \( A \) of a double topological space \((X, \tau)\), we mean a collection \( C = \{G_j : j \in J\}\) of double open subsets of \( X\) such that \( A \subseteq \bigcup\{G_j : j \in J\}\) then we say that \( C\) covers \( A\). In particular, \( A\) collection \( C\) is said to be an open cover of the space \( X\) iff \( X = \bigcup\{(G^1_j, G^2_j) : j \in J\}\) of double open subsets of \( X\).

4.2. Definition. A double-set \( A\) of DTS \((X, \tau)\) is said to be double compact set iff every double sub cover, that is iff for every collection \( \{G_j : j \in J\}\) of DOS's for which \( A \subseteq \{G_j : j \in J\}\) for \( A = (A_1, A_2)\) such that \( (A_1, A_2) \subseteq (G^1_{j_1}, G^2_{j_2}) \cup \ldots \cup (G^1_{j_n}, G^2_{j_n})\).

4.3. Definition. Let \((X, \tau)\) be DTS and let \( N \in X\). A double set \( N\) of \( X\) is said to be \( \tau\)-nhd of \( x\) iff there exists \( \tau\)-DOS, \( G\) such that \( x \in G \subseteq N\), similarly \( N\) is called a \( \tau\)-double nhd of \( A \subseteq X\) iff there exists an DOS, \( G\) such that \( A \subseteq G \subseteq N\).

4.4. Definition. [3] The DTS \((X, \tau)\) is called pairwise \( T_2\) if given \( x \neq y \in X\) there exists \( G, H \in \tau\) satisfying \( x \in G, y \in H\) and \( G \subseteq (\overline{H})\).

4.5. Proposition. If \((X, \tau)\) is pairwise \( T_2\) then every double compact set is double closed set.

Proof: We shall show that \( \overline{G} \in \tau\) is double open set. Let \( \overline{\mathcal{G}} = (G_1, G_2)\). Since \( X\) is \( T_2\) then for.

Then \( p \in G_2, y \notin H_2 \Rightarrow y \notin H^c_2, \quad G_2 \cap H^c_2 = \phi\)

\( \exists \) double open nhds of \( p, y\) such that \( \mu(p) \cap N(y) = \phi\)

Now the collection \( \{\mu(p) : p \in G_2\} \) double open cover of \( G_2\)
\( G \) is compact then \( \{G_2 \subset \bigcup \mu(p_i)\} \).

let \( M = \bigcup \mu(p_i), N = \bigcap \nu(y_i) \) then \( N \) is double open nhd of \( y_i \)

We claim that \( M \cap N = \emptyset \),
\( z \in \mu \Rightarrow z \in \mu(p_i) \Rightarrow z \notin N(y_i) \Rightarrow z \notin N \), thus \( M \cap N = \emptyset \).

Since \( G_2 \subset M \), then \( G_2 \cap N = \emptyset \Rightarrow N \subset G_2 \Rightarrow N \subset \overline{G} \), this shows that \( \overline{G} \) contains a nhd of each of its point and so \( \overline{G} \) is DOS otherwise \( G \) is DCS.

4.6. **Proposition.** Let \( A \) and \( B \) be disjoint double compact subsets of a DTS \((X, \tau)\)

Then there exists disjoint DOS's \( G \) and \( H \) such that \( A \subset G \) and \( B \subset H \).

**Proof:** First, let \( x \in A \) be fixed. Since \( X \) is pairwise \( T_2 \) and \( x \notin B \), for each \( y \in B \), \( A \subset (\bigcup B) \). (clearly \( x \in A_1 \), \( y \in B_2 \) \( \Rightarrow y \in B_2^\circ \) for \( A = (A_1, A_2), B = (B_1, B_2) \))

There exist DOS's \( G_y \) and \( H_y \) such that \( x \in G_y \) and \( y \in H_y \). The collection \{\( H_y : y \in B \)\} is a double open cover of \( B \). Since \( B \) is double compact subspace of \( X \), there exist finitely many points \( y_1, y_2, \ldots, y_n \) of \( B \) such that \( B \subset \{H_y : i = 1, 2, \ldots, n\} \), \( (B_1, B_2) \subset \{(H_1, H_2) : i = 1, 2, \ldots, n\} \)

let \( G_x = \bigcap \{G_y : i = 1, 2, \ldots, n\} = \bigcap \{(G^1_y, G^2_y) : i = 1, 2, \ldots, n\} \)

\( H_x = \bigcup \{(H^1_y, H^2_y) : i = 1, 2, \ldots, n\} \) then \( G_x, H_x \) are disjoint open sets such that \( x \in G_x \) and \( B \subset H_x \).

now let \( x \in A \) be arbitrary and let \( G_x \) and \( H_x \) be as constructed above, then evidently the collection \{\( G_x : x \in A \)\} is a double open cover of \( A \). Since \( A \) is a double compact subspace of \( X \). There exist finitely many points, \( x_1, x_2, \ldots, x_m \) such that \( A \subset \bigcup \{G_{x_i} : i = 1, 2, \ldots, m\} \), let \( G = \bigcup \{G_{x_i} : i = 1, 2, \ldots, m\} \) and \( H = \bigcap \{H_{x_i} : i = 1, 2, \ldots, m\} \) then \( G \) and \( H \) are disjoint double open sets such that

\( A \subset G \) and \( B \subset H \).
4.7. **Definition.** [3] The DTS \( (X, \tau) \) is called pairwise \( T_1 \) if given \( x \neq y \in X \) there exists

\[ G \in \tau \text{ with } x \in G, y \not\in G, \text{ and there exists } H \in \tau \text{ with } y \in H, x \not\in H. \]

4.8. **Definition.** [6] The DTS \( (X, \tau) \) is called pairwise \( T_3 \) if \( \forall \) DCS \( A \in \tau, a \in \text{int } A \text{ in } X \) there exists \( G, H \in \tau \) satisfying \( a \in G, a \not\in H, \ A \subseteq H \) and \( G \subseteq (\overline{H}). \)

4.9. **Proposition.** The DTS \( (X, \tau) \) is called pairwise \( T_1 \) iff every singleton double set \( \{x\} \) of \( X \) is DCS .

**Proof:** \( \iff \) Let every singleton double set \( \{x\} \) of \( X \) be DCS to show that the space is \( T_1 \). Let \( x, y \) be any two disjoint double point of \( X \), then \( \{x\} \) is a DOS which contain \( y \).

Similarly \( \{y\} \) is a DOS which contain \( x \) but does not contain \( y \). Hence \( (X, \tau) \) is pairwise \( T_1 \).

\( \Rightarrow \) Let the space be pairwise \( T_1 \) and let \( x \) be any point of \( X \), we want to show that \( \{x\} \) is DCS, that to show \( X-\{x\} \) is DOS. Let \( y \in X-\{x\} \) then \( x \neq y \) since \( X \) is pairwise \( T_1 \).

There exist an open \( G \) such that \( y \in G \) but \( x \not\in G \). It follows that \( y \in G \subseteq X-\{x\} \).

\( \therefore x \in X-\{y\} \) this means \( x \in X-\{(\phi,\{y\})\} \implies x \not\in X\{y\}, x \in X-\{y\} \) and \( y \not\in X-\{y\} \) then there exists a DOS \( H_y \) such that \( x \in H_y \) but \( y \not\in H_y \), it follows that \( x \in H_y \subseteq X-\{y\} \). Hence \( X-\{y\} \) is DOS. Accordingly \( \{x\} \) is DCS .

4.10. **Proposition.** For a DTS \( (X, \tau) \) pairwise \( T_3 \) is pairwise \( T_1 \).

**Proof:** Let \( \text{ DTS } (X, \tau) \) be pairwise \( T_3 \), we have \( G=(A,B), \ H=(C,D) \in \tau \)
with \( x \in G, x \not\in H \) and \( G \subseteq (\overline{H}) \) i.e \( A \subseteq D \), take \( x \neq y \) in \( X \) and \( y \in H \)

\( \Rightarrow y \not\in D \Rightarrow y \not\in A \Rightarrow y \not\in G \) Then \( (X, \tau) \) is \( T_1 \).
4.11. Proposition. For a DTS \((X, \tau)\) pairwise \(T_3\) is pairwise \(T_2\).

Proof: Let DTS \((X, \tau)\) be pairwise \(T_3\). Take \(x \neq y\) in \(X\). Since \(X\) is pairwise \(T_1\), then there exist \(G \in \tau\) with \(x \notin G\) and \(H \in \tau\) with \(y \in H\) and \(x \notin H\), and since \(X\) is pairwise regular there exist \(G, H \in \tau\) such that \(x \notin G\) and \(x \notin H\), \(G \subseteq (\sim H)\) so that \(x \notin G\), \(y \in H\) and \(G \subseteq (\sim H)\). Accordingly \((X, \tau)\) is \(T_2\). \(\blacksquare\)

References


