Thermal Effect on Viscoelastic Stress Analysis for Incompressible Material

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Abstract

This research permits to use a numerical method for Viscoelastic stress analysis , by using a three parameter model for linear incompressible material and assuming thermorehologically simple material with lower Poison's ratio. The profit of the method is included in this research using finite element method. Thermal effect is indicated here using WLF equation , and this increase the computational efficiency by taking a large time steps. Examples is indicated and comparison of the results with a software program (NASTRAN) and another references which used another methods for solutions will be done , as well as, the efficiency of the method will be discussed.

الخلاصة

هذا البحث يسمح باستعمال اسلوب لتحليل الاجهادات للمواد Viscoelastic باستعمال النموذج three parameter للمواد الخطية غير القابلة للانضغاط على فرض أن المادة هي من نوع thermoreohologically البسيط مع قيم منخفضة نسبيا من v . تم استخدام اسلوب حل عددي finite element لحل المعادلات بعد تطوير الاسلوب ليشمل المواد Viscoelastic كما ويمكن اخذ خطوات زمن اكبر(Δt)مما يزيد فعالية العملية الحسابية. يسمح هذا البحث بادخال واخذ التأثير الحراري بنظر الاعتبار عن طريق المعادلة WLF .
تم تطبيق الاسلوب على امثله ومقارنتها بأساليب حل أخرى مع مقارنة بالبرنامج الخاص بتحليل التراكيب (NASTRAN) وذلك لغرض مقارنة النتائج والتأكد من كفاءة الاسلوب. كما يتضمن البحث مناقشة فعالية اسلوب الحل.

1-Introduction

In the present work a simple and efficient special purpose solution for Viscoelastic stress analysis of incompressible solids is developed using two – dimensional plane strain isoparametric elements.

Some assumptions have been developed for this case [1] :

1- linear Viscoelastic with stress – strain relation and integral transform techniques .
2- bulk modulus is constant in time.
3- homogeneous and isotropic material.

Lakes[2] has used a method for measuring Viscoelastic properties of solids. He showed that the time – temperature superposition principle is not appropriate for all materials i.e. not all materials are thermo- rheologically simple.

Lee. and Rogers [3] solved stress analysis problems for linear Viscoelastic materials on basis of integral operator stress – strain relations by using the method of simple finite – difference numerical integration . They recommend to take the integral from 0 to t and consider the material is undisturbed for $t<0$.

Taylor and Pister [4] developed a computational algorithm for the solution of uncoupled , quasi – static boundary value problem for a linear Viscoelastic solids undergoing thermal mechanical deformation, they showed that the stresses at a high temperature will decrease faster than at a lower temperature.

Ozza and Mccable [5] developed a method for measuring long – term creep and relaxation testing for viscoelastic material . They made studies for materials which do not obey time – temperature superposition .

Thermal effect is important in viscoelasticity, there is really drastic temperature effect of a kind that has never been considered in classical theories that the mean relaxation time of material depends very strongly on temperature[6] .

In observing the mechanical viscoelastic behavior of solids, it is common experience that the stress at a particle depends on both the localized motion of the solid, as well as, the temperature[7].

The conduction equation is assumed to be unaffected by the deformation and therefore solved separately , but simultaneously , with the mechanical field problem, assuming thermorheologically simple material [8].

The influence of temperature can be characterized conveniently by defining a reduced time τ which incorporates the temperature – dependents time scale factor, so that in term of τ , the laws of thermal viscoelastic are applied correspondingly to some chosen temperature Ts which called the reference temperature.

A computational method based on finite element technique with using isoparametric element and local coordinate (natural coordinate) will be applied [9].viscoelastic solution is obtained using Laplace transform technique[10].

The Laplace transform technique is not directly applicable for the problems of non – homogenous transient temperature distributions, to circumvent this problem, conditions of constant temperature over time increments are imposed and the correspondence principle is applied on an incremental basis.

A discussion of mechanical constitutive Eqn for viscoelastic solids undergoing (small or large) deformations and subjected to temperature change are indicated.

As an application of the method , a problem which studied by Zienkwicz [11] is examined, as will as, a checking with a software program (NASTRAN) is done in order to know the efficiency of the procedure.

2-Material and method

2-1- Material representation

For a viscoelastic material , a model can be used to relate components of strain to components of stress.

For incompressible Viscoelastic solid material , the more convenient famous model to represent is called “three parameter model”[12], generally, this model used to represent most standard linear Viscoelastic solids as shown in Fig1.

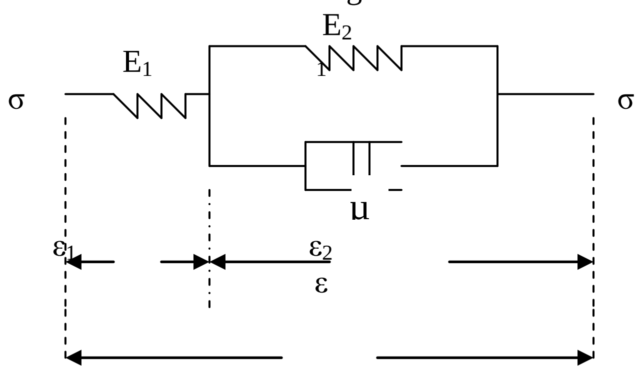


Fig 1. The three parameter model

This model which is consistently used in subsequent applications, it is useful to establish systematically its relaxation modules G and creep compliance J using Laplace transform techniques [13] as following in table 1 :

Table1 : The Laplace transform technique

Constitutive equation.	**Laplace transform**
$\varepsilon_1 = \dfrac{\sigma}{E_1}$	$\bar{\varepsilon}_1 = \dfrac{\bar{\sigma}}{E_1}$
$\varepsilon_2 = \dfrac{\sigma'}{E_2}; \mu \dfrac{d\varepsilon_2}{dt} = \sigma''$	$\bar{\varepsilon}_2 = \dfrac{\bar{\sigma}'}{E_2}; s\mu\, \bar{\varepsilon}_2 = \bar{\sigma}''$
$\sigma' + \sigma'' = \sigma$	$\bar{\sigma}' + \bar{\sigma}'' = \bar{\sigma}$
$E_2\varepsilon_2 + \mu \dfrac{d\varepsilon_2}{dt} = \sigma$	$(E_2 + s\mu)\bar{\varepsilon}_2 = \bar{\sigma}$
$\varepsilon_1 + \varepsilon_2 = \varepsilon$	$\bar{\sigma}\left(\dfrac{1}{E_1} + \dfrac{1}{E_2 + s\mu}\right) = \bar{\varepsilon}$

$$\bar{\varepsilon}(t) = \tilde{J}\bar{\sigma} \quad \ldots 1$$

$$\tilde{J}(s) = \frac{1}{E_1} + \frac{1}{E_2 + s\mu} \quad \ldots 2$$

$$\tilde{G}(s) = \frac{1}{\tilde{J}(s)} = E_1 \frac{s + E_2/\mu}{s + (1/\mu)(E_1 + E_2)} \quad \ldots\ldots 3$$

Where: ε -strian σ -stress E -elasticity modulus μ -viscosity s- Laplace transform factor.

Applying the inverse Laplace transform and simplifying Egn 2,3 can be reduced to :

$$J(t) = \left[\frac{E_1 + E_2}{E_1E_2} - \frac{1}{E_2} EXP\left(-\frac{E_2}{\mu}t\right)\right] \quad \ldots 4$$

$$G(t) = \left[\frac{E_1E_2}{E_1 + E_2} - E_1 EXP\left(-\frac{E_1 + E_2}{\mu}t\right)\right] \quad \ldots 5$$

2-2-Method of work (finite element method)

The displacement based finite element method is one such numerical procedure ,the effectiveness of the method is due to its conceptual simplicity, assuming that the nodal point displacement of the finite element mesh completely specify the displacement in the body.

This finite element technique , which has demonstrated to provide an excellent analysis method for elastic case , has been extended to provide analysis capability for the Viscoelastic case in this research .

The relation of stress- strain for plane strain case are [14] :

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - v(\sigma_{yy} + \sigma_{zz})) \quad ....6$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - v(\sigma_{xx} + \sigma_{zz})) \quad ....7$$

$$\varepsilon_{xy} = \frac{2(1+v)}{E}\sigma_{xy} \quad ....8$$

But : $$\upsilon = \frac{1 - \frac{2G}{K}}{2 + \frac{2G}{K}} \quad ....9$$

$$E = \frac{9K}{1 + \frac{3K}{G}} \quad ......10$$

Then we can obtain from Eqns 6 ,7, 8 the stress matrix {D}(matrix of properties) in term of relaxation G and bulk K module.

The global coordinate {X} of the node in terms of local coordinate (ξ, η) and displacement field $\{\delta\}$ in isoparametric element is [9] :

$$\{X\} = [N]\{X_i\} = \begin{vmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{vmatrix} \quad ......11$$

$$\{\delta\} = [N]\{\delta_i\} = \sum_{i=1}^{n} \delta_i N_i \quad ....12$$

{N} is a matrix of shape function, which is a function of local coordinate ξ and η.

By differentiation of shape function with respect to global coordinate we can obtain strain quantities. This can be done by a transformation using Jacobian matrix{J} which can be obtained by differentiate Eqn 11 using chain rule.

$$\{J\} = \sum_{i=1}^{n} \begin{pmatrix} \frac{\partial Ni}{\partial \eta} xi & \frac{\partial Ni}{\partial \eta} yi \\ \frac{\partial Ni}{\partial \xi} xi & \frac{\partial Ni}{\partial \xi} yi \end{pmatrix} \quad .....13$$

Then local coordinates can be obtained as:

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} dx \\ dy \end{Bmatrix} \quad .....14$$

For plane strain case the relation between strain and displacement is [14]

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad ..... 15$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad .... 16$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad .... 17$$

Then the strain matrix {B} is obtained by writing Eqns 15,16,17 in terms of matrix notation and using the following relations:

$$\frac{\partial u}{\partial x} = \sum_{i=1}^{n} \frac{\partial Ni}{\partial x} ui \quad .... 18$$

$$\frac{\partial v}{\partial y} = \sum_{i=1}^{n} \frac{\partial Ni}{\partial y} vi \quad .... 19$$

It is incorrect to vary only stress matrix {D} with time (the Quasi – static solution) since properties of viscoelastic material varies with time, but it is convenient to differentiate this matrix with respect to time depending on the superposition theory of linear viscoelasticity :

So that :

$$[\frac{\partial D}{\partial t'}] = -\frac{\partial G(\tau - \tau')}{\partial t'} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \{\bar{D}\} \quad .... 20$$

$\tau - \tau^{-}$ is the current and past shifted time respectively which can be calculated from WLF Eqn which have the formula [15]:

$$\log a_T = -\frac{c_1(T - T_S)}{c_2 + T - T_S} \quad .... 21$$

Where Ts is the reference temperature (which represent material's specific constant for the position of the glass transition of the material).

C_1 , C_2 are constants relating to the choice of reference temperature.

From the chosen model in Fig.1 and for the incompressible linear viscoelastic material undergoes environmental temperature change, the total stress will be:

$$\sigma_{total} = \sigma_{elastic} + \sigma_{viscoelastic} + \sigma_{thermal} \qquad \text{----------} \quad 22$$

$$\{\sigma(t)\} = [D]\{\varepsilon(t)\} + \int_0^t [D]\{\varepsilon(t)\}dt - 3\alpha K[T(x_1 t) - T(x_1 0) \qquad \text{----------} \quad 23$$

α - thermal expansion which is constant in time .

By minimizing the equation of potential energy we can solve Eqn.- 23

The minimum potential energy M can be expressed as [16]:

$$M = \frac{1}{2}\int_v [\sigma(t)]\,\varepsilon^T(t)dv - \int_v [\delta]^T Fv \ dv - \int_s [\delta]^T Fs\,ds \qquad \text{-----------} \quad 24$$

F_v : is the body force per unit volume

F_s : is the load of surface traction

By substituting Eqns 23 , 12 into Eqn 24 and minimization with respect to nodal displacements the total potential energy can be written as :

$$\frac{\partial M}{\partial[\delta^e]^T} = 0 = \int_{ve} B^T DBdv\{\delta^e\} + \int_{ve} B^T D\left[\int_0^t \{\varepsilon(t)\}dt\right]dv - \int_{ve} N^T Fvdv - \int_{se} N^T Fsds -$$
$$3K\alpha\int_{ve} B^T(T(X,t) - T(X,0))dv \qquad \text{-----------} \quad 25$$

Solving Eqn 25 will give the values of displacements for all nodes in the structure of interest.

Then stresses can be obtained by solving Eqn 23.

For incompressibility conditions it is more convenient to separate the stress matrix {D} into two components (shear and bulk) as[10] :

$$[D] = [D]^s + [D]^b \qquad \text{---------} \quad 26$$

And by applying a selective integration procedure , which is third order Gauss rule for shear components and second order Gauss rule for bulk components.

This will make some equilibrium between shear and bulk components.

3- Result and Discussion

The first step is obviously to test the rate of convergence and the other features of the process. The process of numerical analysis described in this research is applied to the problem in Fig 2 which was solved by Zienkiewicz[11] .

The problem is a cylinder of Viscoelastic material surrounded by a case of steel and subjected to an internal pressure suddenly applied at t = 0 and maintained thereafter at a magnitude P_o.

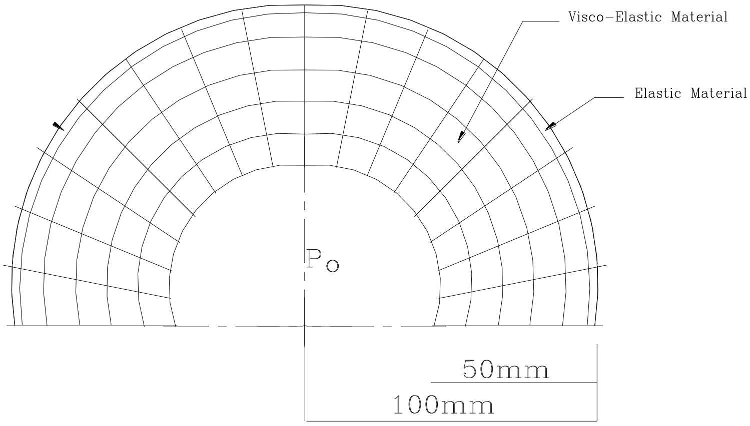


Fig.2 Reinforced Cylinder

The Viscoelastic material is assumed to be isotropic with the following properties:

$$\frac{1}{Kcreep} = 0$$

$G_o = 2584.125 * 10^5$ N/m^2

$K = 6891 * 10^5$ N/m^2

$G(t) = 2584.125*10^5 + 3*10^5 e^{-0.57\tau}$

T_s=75 C

The properties of steel case is taken as :

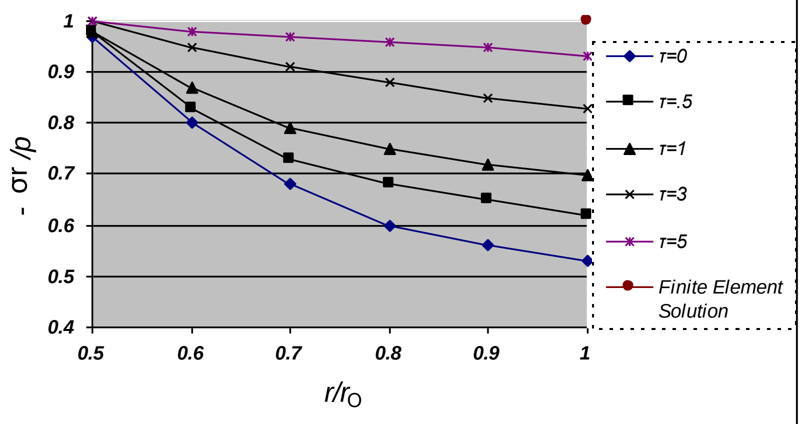
E= 206.73 GPa

$\nu = 0.3015$

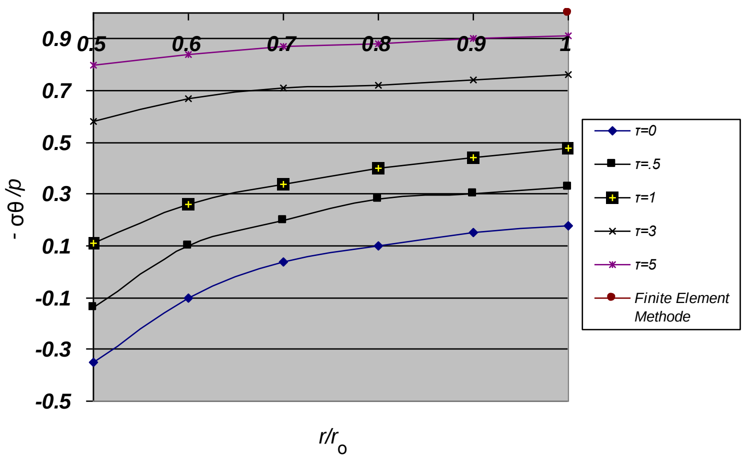
The results of applying the method of Viscoelastic technique is compare with the solution by Zienckiewicz in Figs 3 and 4, where the variation of radial and circumferential stresses with time is shown. There are very small differences from the values of the solution of reference[11] .

The points from finite element solution are obtained by averaging stresses across the element boundaries .

The curves presented are obtained by taking a time step of Δ t = 0.5



________Solution by Zienckwicks
Fig3 Reinforced Cylinder Under Internal Pressure. Variation of Radial Stress



___Solution by Zienckwick
Fig4 Reinforced Cylinder Under Internal Pressure. Variation of Tangential Stress.

Another comparison is made by comparing the results with a software NASTRAN (NASA structural analysis) as shown in table 2:

Table 2 : Comparison of results with software NASTRN at τ = 1

r/r_o	Viscoelastic Solution FEM $\frac{\sigma_r}{P}$	Software NASTRAN $\frac{\sigma_r}{P}$	Viscoelastic Solution FEM $\frac{\sigma_\theta}{P}$	Software NASTRAN $\frac{\sigma_\theta}{P}$
0.5	0.96	0.99	0.12	0.14
0.6	0.87	0.9	0.24	0.25
0.7	0.79	0.81	0.33	0.34
0.8	0.77	0.79	0.41	0.42
0.9	0.75	0.77	0.45	0.456
1	0.74	0.75	0.48	0.492

The main computational advantage of this method over others lies in the fact that larger time steps can be taken. For example in reference[11] to obtain the curve in Fig3 at τ = 3,a thirty time steps is used, and this required thirty solutions of a set of equations, the same curve is obtained by the method of this research using six time steps, as well as, the method can cover the environmental phenomenon. To capture the transient phenomenon for temperature displacements and applied loads , the time steps was taken small enough. The using of the shifted time τ in Viscoelastic solution enables us to include the thermal effect by using WLF Eqn , as well as, using isoparametric element with local coordinates (ξ, η) enable us to use an element with curvilinear shape and cover the change in displacements with time.

Using of selective integration and separation of bulk from shear components will improve the values of results for all permissible values of Poissons ratio(ν).

4-Conclusion and Recommendations

4-1- Conclusion: the present method of Viscoelastic problem permits the use of a general Viscoelastic material representation and allows to take the thermal effects into accounts .

This permits to increase the computational efficiency by taking a large time steps than with step by step process.

4-2- Recommendations

- Extending the solution to thermo mechanical coupling case.
- Extending the case for non-linear solution .

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$$\frac{/\ \mu_2}{E_1 + E_2)}$$

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