Gosper's Algorithm and Hypergeometric Solutions

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Abstract

In this paper, we study Gosper’s algorithm where we use Petkovšek’s technique to give a derivation for Gosper’s algorithm. We show that the least common multiplier can be used to give two simpler algebraically motivated approaches to find hypergeometric solutions of linear recurrences with the additional restriction that the leading and trailing coefficients are constant. In the second approach we use the universal denominator idea. The main result of these approaches that finding hypergeometric solutions reduces to finding polynomial solutions.

Keywords: Gosper's algorithm, least common multiplier, hypergeometric solution, rational solution, universal denominator.

1. Introduction

Let \( N \) be the set of nonnegative integers, \( K \) be a field of characteristic zero, \( K(n) \) be the field of rational functions over \( K \), \( K[n] \) be the ring of polynomials over \( K \), \( \deg(p) \) denotes the polynomial degree in \( n \) of any \( p \in K[n] \), \( \text{lc}(p(n)) \) be the leading coefficient of any \( p \in K[n] \), \( E \) be the shift operator on \( K[n] \), i.e., \((Ep)(n) = p(n + 1)\) for any \( p \in K[n] \), \( \text{lcm} \) be the least common multiplier for polynomials, \( \text{gcd} \) be the greatest common divisor for polynomials. As usual, we assume that subject to normalization the \( \text{gcd} \) of two polynomials always takes a value as a monic polynomial, namely, polynomials with the leading coefficient being 1.

A nonzero term \( t_n \) is called a hypergeometric term over \( K \) if the consecutive term ratio

\[
\frac{t_{n+1}}{t_n} = r(n).
\]
Husam L. Saad and Masar E. Mohammed

Gosper’s Algorithm and ....

is a rational function. If \( r(n) = a(n)/b(n) \), where \( a(n), b(n) \in K[n] \) then the function \( a(n)/b(n) \) is called a rational representation of the rational function \( r(n) \).

Gosper’s algorithm belongs to the standard methods implemented in most computer algebra systems. It has been extensively studied and widely used to verify hypergeometric identities, see [Chen et al (2005), Chen et al (2008), Gerhard(1998), Gosper (1978), Koepf(1998), Milekoric(2005), Koepf(1996), Lisoněk(1994), Qing-Hu Hou(2004)]. Now we give a historical survey of Gosper’s algorithm and hypergeometric solutions. [Gosper (1978)] clarified that any rational function \( r(n) \) can be written in the following form, called Gosper representation

\[
r(n) = \frac{a(n)}{b(n)} \cdot \frac{c(n+1)}{c(n)},
\]

(1.1)

where \( a, b \) and \( c \) are polynomials over \( K \) and

\[
\gcd(a(n), b(n+h)) = 1 \quad \text{for all nonnegative integer } h. \quad \text{(1.2)}
\]

[Petkovšek (1992)] perceives that the Gosper representation becomes unique which is called the Gosper-Petkovšek representation or GP representation, for abbreviate, If \( b \) and \( c \) are monic polynomials such that

\[
\gcd(a(n), c(n)) = 1,
\]

\[
\gcd(b(n), c(n+1)) = 1.
\]

[Petkovšek (1992)] derived the algorithm Hyper to find all hypergeometric solutions of the recurrence

\[
\sum_{i=0}^{d} p_i(n) \cdot z_{n+i} = 0,
\]

where \( p_0(n), p_1(n), ..., p_d(n) = p(n) \in K[n] \) are given polynomials. The algorithm Hyper can be used to find all hypergeometric solutions of the recurrence

\[
\sum_{i=0}^{d} p_i(n) \cdot z_{n+i} = t_n,
\]

(1.3)

where \( t_n \) is a given hypergeometric term such that \( p_0(n) \) and \( p_d(n) \) are constants. [Petkovšek (1994)] gave a derivation for Gosper’s algorithm and a derivation to find all hypergeometric solutions of the recurrence (1.3) with the additional restriction that \( p_0(n) \) and \( p_d(n) \) are constants. [Chen and Saad(2005)] presented a simplified version for Gosper’s algorithm by using GP representation which is similar to the version of Paule and Strehl. They found all hypergeometric solutions of (1.3) such that \( p_0(n) \) and \( p_d(n) \) are constants.

Many approaches, even ours, for generalizing Gosper’s algorithm can be reduced to find rational solutions. So in this paper we need to mention to the rational solutions \( y(n) \) for the linear difference equation
where \( p_0(n), p_1(n), ..., p_d(n), p(n) \in K[n] \) are given polynomials such that \( p_0(n) \) and \( p_d(n) \neq 0 \). A polynomial \( g(n) \in K[n] \) is called a universal denominator for (1.4) if for every solution \( y(n) \in K(n) \) for (1.4) there exists \( f(n) \in K[n] \) such that \( y(n) = f(n)/g(n) \). In order to find rational solutions of (1.4), first we need to find the universal denominator for the equation (1.4) and then find the polynomial solution of the resulting equation.

2. A derivation of Gosper's Algorithm

[Petkovšek (1994)] presented a derivation to find hypergeometric solutions of the linear recurrence (1.3) with the additional restriction that the leading and trailing coefficients are constant. In this section, we use Petkovšek’s technique to give a derivation for Gosper’s algorithm.

Given a hypergeometric term \( t_n \) and suppose that there exists a hypergeometric solution \( z_n \) satisfying

\[
z_{n+1} - z_n = t_n.
\]

Let \( y(n) = z_n/t_n \). By using (2.1) we get

\[
y(n) = \frac{z_n}{t_n} = \frac{z_n}{z_{n+1} - z_n} = \frac{1}{z_n - z_{n+1}}.
\]

Hence \( y(n) \) is an unknown rational function of \( n \). Substitute \( y(n)t_n \) instead of \( z_n \) in (2.1), we obtain

\[
r(n)y(n+1) - y(n) = 1,
\]

where \( r(n) = t_{n+1}/t_n \) is a rational function of \( n \). Thus the problem of finding hypergeometric solution of (2.1) is reduced to the problem of finding rational solution of (2.2). Let us express \( r(n) \) in terms of their Gosper representation as in (1.1). Let

[Chen et al (2008)] found a convergence property for the gcd of the raising factorial and falling factorial. Based on this property, they presented an approach to compute the universal denominator that appears in Gosper's algorithm. They found all hypergeometric solutions of the recurrence (1.3),[Paule(1995)].
\[ k_\circ = \text{dis}(a(n-1), b(n)) = \max \{k \in \mathbb{N} : \deg \gcd(a(n-1), b(n+k)) \geq 1\}, \quad (2.3) \]

and
\[ y(n) = \frac{f(n)}{g(n)c(n)}, \]

such that \( \gcd(f(n), g(n)) = 1 \). Substitute this together with (1.1) into (2.2), we get
\[ a(n) g(n) f(n + 1) - b(n) f(n) g(n + 1) = b(n) c(n) g(n) g(n + 1). \]

From this equation we get
\[ g(n) | b(n) g(n+1) \quad \text{and} \quad g(n) | a(n-1) g(n-1). \]

Using these two relations repeatedly, we obtain
\[ g(n) | b(n) b(n+1) \ldots b(n+k-1) g(n+k), \]
\[ g(n) | a(n-1) a(n-2) \ldots a(n-k) g(n-k), \]

for large enough \( k \). Since \( K \) has characteristic zero then
\[ \gcd(g(n), g(n+k)) = \gcd(g(n), g(n-k)) = 1, \]
for all large enough \( k \). It follows that
\[ g(n) | b(n) b(n+1) \ldots b(n+k-1), \]
\[ g(n) | a(n-1) a(n-2) \ldots a(n-k), \]
for all large enough \( k \). Therefore
\[ g(n) | \gcd(a(n-1)a(n-2) \ldots a(n-k) b(n+1) \ldots b(n+k-1)) \]

When \( k \) goes to infinity we find
\[ g(n) | \gcd(a(n-1)a(n-2) \ldots a(n-k) b(n+1) \ldots b(n+k)) \]

Since \( \gcd(a(n), b(n+h)) = 1 \quad \forall h \in \mathbb{N} \), it follows that \( g(n) \) is a constant. Then we can write \( y(n) = q(n)/c(n) \), where \( q(n) \) is unknown polynomial. Substitute (1.1) and \( y(n) = q(n)/c(n) \) into (2.2) gives
\[ a(n) q(n+1) = b(n) \big(q(n) + c(n)\big). \quad (2.4) \]

This shows that \( b(n) \) divides \( q(n+1) \). Let \( x(n) = q(n)/b(n-1) \) and substitute this in (2.4) to obtain
\[ a(n) x(n+1) - b(n-1) x(n) = c(n). \quad (2.5) \]

Therefore finding hypergeometric solutions of (2.1) is equivalent to finding polynomial solutions of (2.5). The correspondence between them is that if \( x(n) \) is a nonzero polynomial solution of (2.5) then
\[ z_n = \frac{b(n-1) x(n)}{c(n)} t_n, \quad (2.6) \]

is a hypergeometric solution of (2.1), and vice versa.
Example 2.1. Let \( t_n = n \left( \frac{n - 1}{2} \right)! \). Note that

\[
\begin{aligned}
    r(n) &= \frac{t_{n+1}}{t_n} = \frac{\left( \frac{n + 1}{2} \right)}{(n + 2)} \frac{(n + 1)!}{n!} = \frac{a(n)}{b(n)} \frac{c(n + 1)}{c(n)}.
    \\
    \text{Hence } \ a(n) &= n + \frac{1}{2}, \ b(n) = n + 2, \ c(n) = n.
\end{aligned}
\]

From equation (2.5) we find

\[
(n + \frac{1}{2}) x(n + 1) - (n + 1) x(n) = n.
\]

The polynomial \( x(n) = 2(n + 1) \) is a solution to this equation. Therefore, by (2.6) we obtain

\[
z_n = 2(n + 1) \left( \frac{n - \frac{1}{2}}{n!} \right)^n.
\]

3. Hypergeometric Solutions for Recurrences of Arbitrary Order

In this section we give an approach to find the hypergeometric solutions of the linear recurrence (1.3) with the additional restriction that the leading and trailing coefficients are constant. Given a hypergeometric term \( t_n \) and the polynomials \( p_o(n), p_1(n), \ldots, p_d(n) \) with the additional constraints that \( p_0(n) \) and \( p_d(n) \) are constant. Suppose that there exists a hypergeometric solution \( z_n \) of (1.3). Let

\[
y(n) = \frac{z_n}{t_n}, \quad \text{by using (1.3) we get}
\]

\[
y(n) = \frac{z_n}{t_n} = \frac{z_n}{\sum_{i=0}^{d} p_i(n) z_{n+i}} = \frac{1}{\sum_{i=0}^{d} p_i(n) \prod_{j=0}^{i-1} z_{n+j+1}}.
\]

Since \( \frac{z_{n+j+1}}{z_{n+j}} \) is a rational function \( \forall \ j = 0, 1, \ldots, i - 1 \), it follows that \( y(n) \) is unknown rational function of \( n \). Substituting \( y(n) t_n \) for \( z_n \) in (1.3), we get

\[
\sum_{i=0}^{d} p_i(n) y(n+i) \prod_{j=0}^{i-1} r(n+j) = 1, \quad (3.1)
\]

where \( r(n) = t_{n+1}/t_n \) is a rational function of \( n \), then the problem of finding hypergeometric solution of (1.3) is reduced to the problem of finding rational solutions of (3.1). Let us express \( r(n) \) in terms of their Gosper representation as in (1.1). Let

\[
y(n) = \frac{f(n)}{g(n)c(n)} \quad \text{such that}
\]

\( \gcd(f(n), g(n)) = 1 \). Substitute this together with (1.1) into (3.1), we obtain

\[
\sum_{i=0}^{d} p_i(n) \frac{f(n+i)}{g(n+i)c(n+i)} \prod_{j=0}^{i-1} \frac{a(n+j)c(n+j+1)}{b(n+j)c(n+j)} = 1.
\]

Multiplying this equation by \( \prod_{j=i}^{d-1} \frac{b(n+j)}{b(n+i)} \), we obtain
\[
\sum_{i=0}^{d} p_i(n) \frac{f(n+i)}{g(n+i)} \prod_{j=0}^{i-1} a(n+j) \prod_{j=i}^{d-1} b(n+j) = c(n) \prod_{j=0}^{d-1} b(n+j) \quad (3.2)
\]

Letting

\[l(n) = \text{lcm}(g(n+1), g(n+2), \ldots, g(n+d)), \quad (3.3)\]

and multiplying equation (3.2) by \(l(n)g(n)\), we obtain

\[
\sum_{i=0}^{d} p_i(n) f(n+i) \frac{l(n)}{g(n+i)} \prod_{j=0}^{i-1} a(n+j) \prod_{j=i}^{d-1} b(n+j) g(n) = c(n)l(n)g(n) \prod_{j=0}^{d-1} b(n+j).
\]

From (3.3), we have the following divisibility conditions

\[g(n+i) \mid l(n) \quad \text{for} \quad i = 1, 2, \ldots, d.\]

Thus \(l(n)/g(n+i)\) are polynomial for \(i = 1, 2, \ldots, d\). From the above equation we obtain

\[g(n) \mid p_\ast(n) \prod_{j=0}^{d-1} b(n+j) \text{lcm}(g(n+1), g(n+2), \ldots, g(n+d)). \quad (3.4)\]

Similarly, multiplying equation (3.2) by \(l(n-1)g(n+d)\) and then substituting \(n-d\) for \(n\), we obtain that

\[g(n) \mid p_\ast(n-d) \prod_{j=1}^{d} b(n-j) \text{lcm}(g(n-1), g(n-2), \ldots, g(n-d)). \quad (3.5)\]

shifting \(n\) by 1 in (3.4) yields

\[g(n+1) \mid p_\ast(n+1) \prod_{j=1}^{d} b(n+j) \text{lcm}(g(n+2), g(n+3), \ldots, g(n+d+1)),\]

substituting this equation in (3.4) we see that \(g(n)\) divides

\[p_\ast(n) \prod_{j=0}^{d-1} b(n+j) \text{lcm}(p_\ast(n+1) \prod_{j=1}^{d} b(n+j) \text{lcm}(g(n+2), g(n+3), \ldots, g(n+d+1)))\]

So we can write

\[g(n) \mid p_\ast(n) p_\ast(n+1) \prod_{j=0}^{d-1} b(n+j) \prod_{j=1}^{d} b(n+j) \text{lcm}(g(n+2), g(n+3), \ldots, g(n+d+1)).\]

By induction we may get for \(k \geq 1\)
\[g(n) \bigg| \prod_{j=0}^{k-1} p_0(n + j) \prod_{j=0}^{d-1} b(n + j) \prod_{j=1}^{d} b(n + j) \cdots \prod_{j=k-1}^{k+d-2} b(n + j) \lcm(g(n + k), g(n + k + 1), \ldots, g(n + k + d - 1)) .\]

It follows that
\[g(n) \bigg| \prod_{j=0}^{k-1} p_0(n + j) \prod_{i=0}^{k-1} \left( \prod_{j=i}^{d+i-1} b(n + j) \right) g(n + k) \cdot g(n + k + 1) \cdots g(n + k + d - 1) .\]

Since \( K \) has characteristic zero, then for large enough \( k \)
\[\gcd(g(n), g(n + j)) = 1 \quad \text{for} \quad j \geq k .\]

It follow that
\[g(n) \bigg| \prod_{j=0}^{k-1} p_0(n + j) \prod_{i=0}^{k-1} \left( \prod_{j=i}^{d+i-1} b(n + j) \right) ,\]
for large enough \( k \). Analogously form (3.5) we can get
\[g(n) \bigg| \prod_{j=0}^{k-1} p_d(n - d - j) \prod_{i=0}^{k-1} \left( \prod_{j=i+1}^{d+i} a(n - j) \right) ,\]
for large enough \( k \). By assumption \( p_0(n) \) and \( p_d(n) \) are constant. Therefore
\[g(n) \bigg| \gcd \left( \prod_{i=0}^{k-1} \left( \prod_{j=i+1}^{d+i} a(n - j) \right), \prod_{i=0}^{k-1} \left( \prod_{j=i}^{d+i-1} b(n + j) \right) \right) .\]

When \( k \) goes to infinity, we obtain
\[g(n) \bigg| \gcd \left( \prod_{i=0}^{k} \left( \prod_{j=i+1}^{d+i} a(n + j) \right), \prod_{i=0}^{k} \left( \prod_{j=i}^{d+i-1} b(n + j) \right) \right) .\]

Since \( \gcd(a(n), b(n + h)) = 1 \quad \forall h \in N , \) it follows that \( g(n) \) is a constant. Then we can write
\[y(n) = q(n) / c(n) , \quad (3.6)\]
where \( q(n) \) is unknown polynomial. Inserting (1.1) and (3.6) into (3.1) gives
\[\sum_{i=0}^{d} p_i(n) q(n + i) \prod_{j=0}^{i-1} a(n + j) \prod_{j=i}^{d-1} b(n + j) = c(n) \prod_{j=0}^{d-1} b(n + j) .\]

Again \( q(n) \) is divisible by \( b(n - 1) \) therefore we finally get
\[
y(n) = \frac{b(n-1)x(n)}{c(n)},
\]
where \(x(n)\) is an unknown polynomial satisfying
\[
\sum_{i=0}^{d} p_i(n)x(n+i) \prod_{j=0}^{i-1} a(n+j) \prod_{j=i-1}^{d-1} b(n+j) = c(n) \prod_{j=0}^{d-1} b(n+j). \quad (3.7)
\]
Therefore finding hypergeometric solutions of (1.3) is equivalent to finding polynomial solutions of (3.7). The correspondence between them is that if \(x(n)\) is a nonzero polynomial solution of (3.7) then
\[
z_n = \frac{b(n-1)x(n)}{c(n)} t_n, \quad (3.8)
\]
is a hypergeometric solution of (1.3), and vice versa.

**Example 3.1.** Find all Hypergeometric Solutions of
\[
z_{n+2} - z_{n+1} + z_n = t_n, \quad (3.9)
\]
where \(t_n = n+1\). Then
\[
r(n) = \frac{t_{n+1}}{t_n} = \frac{n+2}{n+1} = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)},
\]
where \(a(n) = 1 = b(n), c(n) = n+1\). From (3.7), \(x(n)\) is a polynomial satisfies
\[
x(n+2) - x(n+1) + x(n) = n+1,
\]
The polynomial solution of this equation is \(x(n) = n\). By (3.8), we have
\[
z_n = \frac{b(n-1)x(n)}{c(n)} t_n = n
\]
is a hypergeometric solution of (1.3)

4. **Another Approach for Hypergeometric Solutions for Recurrences of Arbitrary Order**

In this section we give another approach to find hypergeometric solutions of the linear recurrence (1.3) with the additional restriction that the leading and trailing coefficients are constant.
Lemma 4.1. Let

\[ L = \max \left\{ k \in \mathbb{N} : \deg \gcd \left( c(n) \prod_{j=0}^{d-1} a(n-d+j), c(n+k) \prod_{j=0}^{d-1} b(n+j+k) \right) \geq 1 \right\}, \]

and let \( k_{\ast} \) be defined as in (2.3). Then \( L = k_{\ast} \).

Proof. Let

\[ g_1(k) = \gcd(c(n)a(n-1), c(n+k)b(n+k)), \]

and

\[ g_2(k) = \gcd \left( c(n) \prod_{j=0}^{d-1} a(n-d+j), c(n+k) \prod_{j=0}^{d-1} b(n+k+j) \right). \]

For any \( k \) we have \( g_1(k) \mid g_2(k) \) hence \( L \geq k_{\ast} \).

Suppose that for an irreducible polynomial \( t(n) \), \( t(n) \mid g_2(L) \), i.e.

\[ t(n) \mid c(n)a(n-d+i) \quad \text{for some} \quad i \in \{0,1,\ldots,d-1\}, \]

and

\[ t(n) \mid c(n+L)b(n+j+L) \quad \text{for some} \quad j \in \{0,1,\ldots,d-1\}. \]

Hence

\[ t(n+d-i-1) \mid c(n+d-i-1)a(n-1), \]

and

\[ t(n+d-i-1) \mid c(n+d-i-1+L)b(n+d-i-1+j+L). \]

It follows that

\[ t(n+d-i-1) \mid \gcd(c(n+d-i-1)a(n-1), c(n+d-i-1+L)b(n+d-i-1+j+L)). \]

Then we have

\[ d-i-1+j+L \leq k_{\ast}, \]

which implies that \( L \leq k_{\ast} \). Hence \( L = k_{\ast} \). \qed
Given a hypergeometric term \( t_n \) and suppose that there exists a hypergeometric solution \( z_n \) satisfying equation (1.3). We immediately start with equation (3.1) because there is no difference at the beginning. We look for a rational function \( y(n) \) satisfying

\[
\sum_{i=0}^{d} p_i(n)y(n+i) \prod_{j=0}^{i-1} r(n+j) = 1,
\]

where \( r(n) = t_{n+1}/t_n \) is a rational function of \( n \), then the problem of finding hypergeometric solution of (1.3) is reduced to the problem of finding rational solutions of (3.1). Let us express \( r(n) \) in terms of their Gosper representation as in (1.1) and let \( y(n) = f(n)/g(n) \) such that \( \gcd(f(n), g(n)) = 1 \). Substitute this together with (1.1) into (3.1), we obtain

\[
\prod_{j=0}^{d-1} \sum_{i=0}^{d} p_i(n)c(n+i) f(n+i) \frac{l(n)}{g(n+i)} \prod_{j=0}^{i-1} a(n+j) \prod_{j=i}^{d-1} b(n+j) g(n) = c(n) \prod_{j=0}^{d-1} b(n+j). \quad (4.1)
\]

Letting

\[
l(n) = \text{lcm}(g(n+1), g(n+2),..., g(n+d)),
\]

and multiplying equation (4.1) by \( l(n) g(n) \), we obtain

\[
\prod_{j=0}^{d-1} \sum_{i=0}^{d} p_i(n)c(n+i) f(n+i) \frac{l(n)}{g(n+i)} \prod_{j=0}^{i-1} a(n+j) \prod_{j=i}^{d-1} b(n+j) g(n) = c(n)l(n) g(n) \prod_{j=0}^{d-1} b(n+j). \quad (4.3)
\]

From (4.2), we have the following divisibility conditions

\[
g(n+i) | l(n) \quad \text{for } i = 1, 2, ..., d.
\]

Thus \( l(n)/g(n+i) \) are polynomial for \( i = 1, 2, ..., d \). Form (4.3) we obtain

\[
g(n) | p_c(n)c(n) \prod_{j=0}^{d-1} b(n+j) \text{lcm}(g(n+1), g(n+2),..., g(n+d)). \quad (4.4)
\]

Similarly, multiplying equation (4.1) by \( l(n-1) g(n+d) \) and then substituting \( n-d \) for \( n \), we obtain that

\[
g(n) | p_{c_d}(n-d)c(n) \prod_{j=1}^{d} a(n-j) \text{lcm}(g(n-1), g(n-2),..., g(n-d)). \quad (4.5)
\]

Shifting \( n \) by \( 1 \) in (4.4) yields

\[
g(n+1) | p_c(n+1)c(n+1) \prod_{j=1}^{d} b(n+j) \text{lcm}(g(n+2), g(n+3),..., g(n+d+1)). \quad (4.6)
\]
Substituting (4.6) in (4.4) we see that $g(n) \mid d$

Ivides

$$p_0(n)c(n)\prod_{j=0}^{d-1} b(n+j) \text{lcm}(p_0(n+1)c(n+1)\prod_{j=1}^{d} b(n+j) \text{lcm}(g(n+2), g(n+3), \ldots, g(n+d+1)), g(n+2), \ldots, g(n+d)).$$

So we can write

$$g(n) \mid p_0(n)p_0(n+1)c(n)c(n+1)\prod_{j=0}^{d-1} b(n+j) \prod_{j=1}^{d} b(n+j) \text{lcm}(g(n+2), g(n+3), \ldots, g(n+d+1)).$$

By induction we may derive for $k \geq 1$

$$g(n) \mid \prod_{j=0}^{k-1} p_0(n+j) \prod_{j=0}^{k-1} c(n+j) \prod_{j=0}^{d-1} b(n+j) \prod_{j=1}^{d} b(n+j) \prod_{j=k-1}^{k+d-2} b(n+j) \text{lcm}(g(n+k), g(n+k+1), \ldots, g(n+k+d-1)).$$

It follows that

$$g(n) \mid \prod_{j=0}^{k-1} p_0(n+j) \prod_{j=0}^{k-1} c(n+j) \prod_{j=0}^{k} (\prod_{j=1}^{d} b(n+j)) g(n+k) \cdot g(n+k+1) \cdots g(n+k+d-1)$$

Since $K$ has characteristic zero, then for large enough $k$ we get

$$\gcd(g(n), g(n+j)) = 1, \quad \text{for } j \geq k.$$  

It follows that

$$g(n) \mid \prod_{j=0}^{k-1} p_0(n+j) \prod_{j=0}^{k-1} c(n+j) \prod_{j=0}^{k} (\prod_{j=1}^{d+i-1} b(n+j)),$$

for large enough $k$. Analogously form (4.5) we get

$$g(n) \mid \prod_{j=0}^{k-1} p_d(n-d-j) \prod_{j=0}^{k-1} c(n-j) \prod_{i=0}^{d} a(n+j),$$

for large enough $k$. By assumption $p_0(n)$ and $p_d(n)$ are constant. Therefore

$$g(n) \mid \gcd\left(\prod_{j=0}^{k-1} c(n-j) \prod_{i=0}^{d} a(n-j), \prod_{j=0}^{k-1} c(n+j) \prod_{j=0}^{d+i-1} b(n+j)\right),$$

i.e.,

25
Husam L. Saad and Masar E. Mohammed

Gosper's Algorithm and ...

\[ g(n) \mid c(n) \gcd \left( \prod_{j=1}^{k-1} c(n-j) \prod_{i=0}^{d+i} a(n-j), \prod_{j=1}^{k-1} c(n+j) \prod_{i=0}^{d+i-1} b(n+j) \right). \]

When \( k \) goes to infinity, we obtain

\[ g(n) \mid c(n) \gcd \left( \prod_{j=1}^{L} c(n-j) \prod_{i=0}^{d+i} a(n-j), \prod_{j=1}^{L} c(n+j) \prod_{i=0}^{d+i-1} b(n+j) \right). \]

From equation (4.1) we get

\[ \sum_{j=0}^{d} p_j(n) c(n+i) f(n+i) \prod_{j=0}^{k-1} g_j(n+i) \prod_{j=0}^{d+i} a(n+j) \prod_{j=0}^{d+i-1} b(n+j) = c(n) \prod_{j=0}^{L} g_j(n+i) \prod_{j=0}^{d+i} b(n+j). \quad (4.7) \]

The next step is simply to set

\[ g(n) = c(n) \gcd \left( \prod_{j=1}^{L} c(n-j) \prod_{i=0}^{d+i} a(n-j), \prod_{j=1}^{L} c(n+j) \prod_{i=0}^{d+i-1} b(n+j) \right). \quad (4.8) \]

Finding hypergeometric solutions of (1.3) is therefore equivalent to finding polynomial solutions of (4.7). The correspondence between

\[ z_n = \frac{f(n)}{g(n)} t_n, \quad (4.9) \]

is a hypergeometric solution of (1.3), and vice versa.

**Example 4.1.** Find all Hypergeometric Solutions of

\[ z_{n+2} - z_{n+1} + z_n = t_n, \quad (4.10) \]

where, \( t_n = n+1. \) Then

\[ r(n) = \frac{t_{n+1}}{t_n} = \frac{n+1}{n} = \frac{a(n)}{c(n)} = \frac{c(n+1)}{b(n)}. \]

Hence \( a(n) = 1 = b(n), \) \( c(n) = n+1. \)

From Lemma 4.1, \( L = 0. \) From (4.8),

\[ g(n) = n+1. \] By (4.7), \( f(n) \) is a polynomial satisfies

\[ f(n+2) - f(n+1) + f(n) = n+1. \]

The polynomial solution of this equation is \( f(n) = n. \) By (4.9), we have

\[ z_n = \frac{f(n)}{g(n)} t_n = n \]

is the hypergeometric solution of (4.10).

**References**


