Bifurcation periodic solutions of nonlinear fourth order differential equation

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Abstract

In this paper bifurcation periodic traveling-wave solutions of the regularized Boussinesq system which model waves in a horizontal water channel in both directions was studied by using a local scheme of Liapunov –Schmidt. Asymptotic representation of bifurcation periodic solutions is given.

Key words: Boussinesq system for water waves, bifurcation solutions, scheme of Liapunov – Schmidt.

Introduction

It is known that many of the nonlinear problems in mathematics and physics can be written in the form of operator equation,

$$f(x,\lambda)=b, \quad x\in O\subset X, \quad b\in Y, \quad \lambda\in R^n. \qquad \dots \ (1)$$

in which f is a smooth Fredholm map of index zero, X, Y Banach spaces and O open subset of X. For these problems, we can use the method of reduction to finite dimensional equation [1],

$$\Theta(\xi,\lambda)=\beta, \quad \xi\in M, \quad \beta\in N, \qquad \dots(2)$$

where M and N are smooth finite dimensional manifolds.

Passing from equation (1) into equation (2) (variant local scheme of Liapunov –Schmidt) with the conditions that equation (2) has all the topological and analytical properties of equation (1) (multiplicity, bifurcation diagram, etc.) dealing with [2],[5],[6],[7].

In spite of that, the theory of bifurcation solutions of nonlinear Fredholm equations has more works in different ways, but for this time there are more problems that need to be solved by using the method of finite dimensional reduction as in this paper. In the theory of polynomials and holomorphics maps [3] used the term of discriminant set which is similar to the term used in the modern theory of bifurcation.

A continuous linear operator $A: E\rightarrow F$, (E,F are Banach spaces) called Fredholm , iff

$$\dim(Ker\ A)<\infty, \qquad \dim(CoKer\ A)<\infty.$$

the number,

$$\dim(Ker\ A)-\dim(CoKer\ A),$$

is called Fredholm index of operator A.

The nonlinear map $f: \Omega\rightarrow F$, where Ω is an open subset of E, is called Fredholm, if $\frac{\partial f(x)}{\partial x}$ is a Fredholm operator, $x\in\Omega$. The index of f is the same index of $\frac{\partial f(x)}{\partial x}$.

The set of all λ in which equation (1) has degenerate solution in O is called the discriminant set.

In [4] the author considered the regularized Boussinesq system,

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x - \frac{1}{6}\eta_{xxt} = 0, \\ &u_t + \eta_x + u u_x - \frac{1}{6} u_{xxt} = 0. \end{aligned} \qquad \ldots \quad (3)$$

which describes approximately the two-dimensional propagation of surface waves in a uniform horizontal channel of length L filled with an irrotational, incompressible, invisid fluid which in its undisturbed state has depth ***h***. The non-dimensional variables $\eta(x, t)$ and $u(x, t)$ represent respectively, the deviation of the water surface from its undisturbed position and the horizontal velocity at water level $\sqrt{\frac{2}{3}}$ ***h***. In his work [4] he found the traveling-wave solution of system (3) in the form $\xi = x - k\,t$, with the initial and boundary conditions ,

$$\begin{aligned} &\eta(0,t) = h_1(t), \qquad \eta(L,t) = h_2(t), \\ &u(0,t) = v_1(t), \qquad u(L,t) = v_2(t), \\ &\eta(x,0) = f_1(x), \qquad u(x,0) = f_2(x). \end{aligned}$$

where k is the speed of the traveling-wave, also he found an exact solution with a single trough. In this paper, i shall study the two-modes of bifurcation periodic traveling-wave solution in the form $\xi = r\,x - k\,t$, where ($v = k/r$ - is the speed of propagation wave). In this case system (3) has the form,

$$\begin{aligned} &\frac{k r^2}{6}\eta'' + r u - k\eta + r\eta u = c_1, \\ &\frac{k r^2}{6} u'' - k u + r\eta + \frac{r}{2}u^2 = c_2. \end{aligned} \qquad \ldots \ (4)$$

where $\eta(x, t) = \eta(\xi)$ and $u(x, t) = u(\xi)$.
In this work i shall assume that $c_1 = c_2 = 0$.

Bifurcation periodic solutions

It is easy to check that system (4) can be written as a single equation,

$$u'''' - \frac{12}{r^2}u'' + \frac{12}{k r}u u'' + \frac{6}{k r}(u')^2 - \frac{36(r - \frac{k^2}{r})}{k^2 r^3}u - \frac{54}{k r^3}u^2 + \frac{18}{k^2 r^2}u^3 = 0. \quad \ldots (5)$$

We note that it is not difficult to find bifurcation periodic traveling-wave solutions of equation (5) for r=1, so i shall not consider this case in this work. Our purpose is to study the bifurcation periodic traveling-wave solutions of equation (5) with period (2π) in the neighborhood of point zero by using local method of Liapunov-Schmidt to reduce into finite dimensional spaces. For this purpose, it is convenient to set the equation (5) in the form of operator equation, that is;

$$f(u,\lambda) := u'''' - \frac{12}{r^2}u'' + \frac{12}{k r}u u'' + \frac{6}{k r}(u')^2 - \frac{36(r - \frac{k^2}{r})}{k^2 r^3}u - \frac{54}{k r^3}u^2 + \frac{18}{k^2 r^2}u^3, \quad \ldots .(6$$

where $\lambda = (r, k)$ and $f : E \to F$ is a nonlinear Fredholm operator, E- Banach space of all continuous differentiable periodic functions of period (2π), F- Banach space of all continuous periodic functions of period (2π). The study of bifurcation solutions of equation (5) is equivalent to the study of bifurcation solutions of operator equation,

$$f(u,\lambda) = 0. \qquad \ldots (7)$$

In the method of Liapunov-Schmidt it is necessary to determine the kernel of linear operator $f_u(0,\lambda)$, because the basis of the null space N is the kernel of the linear operator $f_u(0,\lambda)$. We can find the kernel of linear operator by linearized equation (7).

Thus, to apply the method of Liapunov-Schmidt i can find the linearized equation (corresponding to the equation (7)),

$$Ah = 0, \quad h \in E,$$

$$A = \frac{\partial f}{\partial u}(0,\lambda) = \frac{d^4}{d\xi^4} - \frac{12}{r^2}\frac{d^2}{d\xi^2} - \frac{36}{k^2 r^2} + \frac{36}{r^4}, \quad \xi \in [0, 2\pi].$$

Note that the operator A is Fredholm and then the point $(0,\lambda)$ is bifurcation point. Periodic solution of linearized equation is of the form,

$$h = p_1 \sin(p\xi) + p_2 \cos(p\xi)$$

which is corresponding to the characteristic equation

$$p^4 r^4 k^2 + 12 k^2 r^2 p^2 - 36 r^2 + 36 k^2 = 0.$$

this equation gives in the *rk*-plane characteristic lines l_p. The point of intersection of two lines is a bifurcation point [8]. In particular, the intersection of the lines l_1 , l_2 is the point (0, 0). Localization parameters,

$\| e_i \| = 1$, i=1,…,4 , i have $a_1 = a_2 = b_1 = b_2 = \sqrt{2}$.

Suppose that $N = Ker\ A = Span\{e_1, e_2, e_3, e_4\}$, then the space E can be decomposed in the direct sum of two subspaces, N and the orthogonal complement to N:

$$E = N \dot{+} E^{\infty-4}, \quad E^{\infty-4} = N^{\perp} \cap E = \{v \in E : v \perp N\}.$$

Similarly, the space F decomposed in the direct sum of two subspaces, N and orthogonal to N:

$$F = N \dot{+} F^{\infty-4}, \quad F^{\infty-4} = N^{\perp} \cap F = \{f \in F : f \perp N\}.$$

In this case, every vector $u \in E$ can be written in the form,

$$u = w + v, \quad w = \sum_{i=1}^{4} \xi_i e_i \in N, \quad N \perp v \in E^{\infty-4}, \quad \xi_i = < w, e_i >.$$

Similarly,

$$f(u,\lambda) = f^{(4)}(u,\lambda) + f^{(\infty-4)}(u,\lambda),$$

$$f^{(4)}(u,\lambda) = \sum_{i=1}^{4} v_i(u,\lambda) e_i \in N, \qquad f^{(\infty-4)}(u,\lambda) \in F^{\infty-4}$$

$$v_i(u,\lambda) = < f(u,\lambda), e_i >.$$

where $f^{(4)}(u,\lambda)$ is the projection of the space F on N and $f^{(\infty-4)}(u,\lambda)$ is the projection of the space F on $F^{\infty-4}$, and

$$f^{(4)}(u,\lambda) = \sum_{i=1}^{4} \frac{< f(u,\lambda), e_i >}{< e_i, e_i >} e_i,$$

$$f^{(\infty-4)}(u,\lambda) = f(u,\lambda) - f^{(4)}(u,\lambda),$$

$$< x, y > = \frac{1}{T}\int_0^T \sum_{i=1}^{4} x_i(t)\, y_i(t)\, dt.$$

Here $T = 2\pi$. Equation (7) can be written in the form,

$$f^{(4)}(w+v,\lambda)=0,$$
$$f^{(\infty-4)}(w+v,\lambda)=0.$$

by implicit function theorem, there exist smooth map $\Phi: N \to E^{\infty-4}$ (depend upon λ), such that

$$f^{(\infty-4)}(w+\Phi(w,\lambda),\lambda)=0,$$

and then i have bifurcation equation in the form,

$$\Theta(\hat{\xi},\lambda)=0,$$

where , $$\Theta(\hat{\xi},\lambda):=f^{(4)}(w+\Phi(w,\lambda),\lambda).$$

equation (6) can be written in the form,

$$f(w+v,\lambda)=Aw+\frac{12}{k\,r}w\,w^{//}+\frac{6}{k\,r}(w^{/})^2-\frac{54}{k\,r^3}w^2+\frac{18}{k^2r^2}\,w^3+\ldots$$

then,

$$f^{(4)}(w+v,\lambda)=\sum_{i=1}^{4}<Aw+\frac{12}{k\,r}w\,w^{//}+\frac{6}{k\,r}(w^{/})^2-\frac{54}{k\,r^3}w^2+\frac{18}{k^2r^2}\,w^3,\,e_i>e_i+\ldots$$

where (< · , · >- scalar product in Hilbert space L_2 ([0, 2 π] , R)) .

After some calculations of $f^{(4)}(w+v,\lambda)$, we have bifurcation equation in the form:

$$\Theta(\hat{\xi},\delta)=\begin{pmatrix}\xi_1(\xi_1^2+\xi_2^2)+2\xi_1(\xi_3^2+\xi_4^2)+q_1(\xi_2\,\xi_3-\xi_1\,\xi_4)+q_2\xi_1\\ \xi_2(\xi_1^2+\xi_2^2)+2\xi_2(\xi_3^2+\xi_4^2)-q_1(\xi_1\,\xi_3+\xi_2\,\xi_4)+q_2\xi_2\\ 2\xi_3(\xi_1^2+\xi_2^2)+\xi_3(\xi_3^2+\xi_4^2)+\alpha_1\xi_1\,\xi_2+\alpha_2\xi_3\\ 2\xi_4(\xi_1^2+\xi_2^2)+\xi_4(\xi_3^2+\xi_4^2)+\beta_1(\xi_1^2-\xi_2^2)+\alpha_2\,\xi_4\end{pmatrix}+\ldots=0 \qquad \ldots(8)$$

where,

$$Ae_i=\tilde{\alpha}_i(\lambda)e_i,\quad \hat{\xi}=(\xi_1,\xi_2,\xi_3,\xi_4),\quad \delta=(q_1,q_2,\alpha_1,\alpha_2,\beta_1).$$

$\tilde{\alpha}_i(\lambda)-$ *smooth spectral function.*

In the complex variables,

$$z_1=\xi_1+i\,\xi_2,\qquad z_2=\xi_3+i\,\xi_4.$$

bifurcation equation can be written in the following form,

$$\begin{aligned}&z_1z_2|z_1|^2+2\,z_1z_2|z_2|^2+\hat{q}_1z_1|z_2|^2+q_2z_1z_2+\ldots=0,\\ &2\,z_2z_1^2|z_1|^2+z_2z_1^2|z_2|^2+\hat{\alpha}_1(z_1^4-|z_1|^4)+\hat{\beta}_1(z_1^4+|z_1|^4)+\alpha_2z_1^2z_2+\ldots=0.\end{aligned} \qquad \ldots(9)$$

where, z_1 , $z_2 \neq 0$ and $\hat{q}_1,\hat{\alpha}_1,\hat{\beta}_1$ are complex numbers.

To solve system (9) it is convenient to consider this system in polar coordinate $\xi_1= r_1 \cos\theta$, $\xi_2= r_1 \sin\theta$, $\xi_3= r_2 \cos\varphi$, $\xi_4= r_2 \sin\varphi$ and then i have the following system,

$$r_1^2 + 2r_2^2 - q_1 r_2 + q_2 + ... = 0,$$
$$2r_1^2 r_2 + r_2^3 + \beta_1 r_1^2 + \alpha_2 r_2 + ... = 0. \qquad ...(10)$$

in which we can determine asymptotic representation of bifurcation periodic solutions. Discriminant set of system (10) locally equivalent in the point zero to the discriminant set of the system,

$$r_1^2 + 2r_2^2 - q_1 r_2 + q_2 = 0,$$
$$2r_1^2 r_2 + r_2^3 + \beta_1 r_1^2 + \alpha_2 r_2 = 0. \qquad ...(11)$$

we note that, it is easy to solve discriminant set of system (11) when β_1=0 but unfortunately, in this work the value of β_1 does not need to be equal to zero, also we need to solve this system with the condition (r_1 , $r_2 > 0$), so in another paper i shall discuss the discriminant set of system (11) for $(r_1, r_2, q_1, q_2, \beta_1, \alpha_2) \in R^6$.

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الخلاصة

في هذا البحث تم دراسة الحلول الدورية للموجة الهاربة لنظام بوسون المنتظم الذي يصف الموجات في قناة ماء افقية في كلا الاتجاهين باستخدام طريقة ليابونوف – شمدت المحلية للتخفيض الى الفضاءات المنتهية . التمثيل المقـــــارب للحلـــــول الدوريـــــة المتفرعـــــة تـــــم ايجـــــاده فـــــي هـــــذا البحـــــث .