The Detour Polynomials of Ladder Graphs

Ali A. Ali & Gashaw A. Mohammed-Saleh
Academic, University of Mosul & College of Science, University of Salahaddin

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Abstract

The detour distance \( D(u,v) \) between two distinct vertices \( u \) and \( v \) of a connected graph \( G \) is the length of a longest \( u-v \) path in \( G \). The detour index \( dd(G) \) of \( G \) is defined by \( \sum_{[u,v]} D(u,v) \), and the detour polynomial of \( G \) is \( D(G;x) = \sum_{[u,v]} x^{D(u,v)} \). The detour indices and detour polynomials of some ladder graphs are obtained in this paper.

Key Words: Detour distance, Detour index, Detour polynomials, Ladder graphs.

1. Introduction

For the definitions of graph concepts and notations see the books [1] and [7].

The detour distance \( D(u,v) \) between two distinct vertices \( u \) and \( v \) in a connected graph \( G \) is the maximum of the lengths of \( u-v \) paths in \( G \) (See [2, 3, 4, 5, 6 and 8]). A \( u-v \) path of length \( D(u,v) \) is called \( u-v \) detour. As with standard distance, the detour distance \( D \) is a metric on the vertex set \( V(G) \) of any connected graph \( G \). That is

1. \( D(u,v) \geq 0 \) for all vertices \( u, v \in V(G) \),
2. \( D(u,v) = 0 \) if and only if \( u = v \),
3. \( D(u,v) = D(v,u) \) for all vertices \( u \) and \( v \) of \( G \), and
4. \( D(u,v) + D(v,w) \geq D(u,w) \) for all vertices \( u, v \) and \( w \) of \( G \).

It is clear that \( D(u,v) = 1 \) if and only if \( uv \) is a bridge of \( G \), and
\( D(u,v) = p(G) - 1 \) if and only if \( G \) contains a hamiltonian \( u-v \) path. Moreover, \( D(u,v) = d(u,v) \) for every two vertices \( u \) and \( v \) of \( G \) if and only if \( G \) is a tree. For other properties of the detour distance see [2 and 5].

The detour eccentricity \( e_D(v) \) of a vertex \( v \) in a connected graph \( G \) is
\[ e_D(v) = \max \{ D(u,v) : u \in V(G) \} \] ... (1.1)

The detour radius \( rad_D(G) \) of a connected graph \( G \) is defined as
\[ rad_D(G) = \min \{ e_D(v) : v \in V(G) \} \] ... (1.2)
while the detour diameter $diam_D(G)$ of $G$ is
\[
diam_D(G) = \max\{e_D(v) : v \in V(G)\}.
\]
In any connected graph $G$, the detour radius and detour diameter are related by the following inequalities[1]:
\[
rad_D(G) \leq diam_D(G) \leq 2rad_D(G).
\]
The detour index $dd(G)$ of a connected graph $G$ is the Wiener index with respect to detour distance, that is
\[
dd(G) = \sum_{u,v} D(u,v),
\]
where the summation is taken over all unordered pairs of vertices $u$ and $v$ of $G$.

The detour distance of a vertex $v$, denoted by $d_D(v)$, is defined by
\[
d_D(v) = \sum_{u \in V(G)} D(u,v).
\]
It is clear that
\[
dd(G) = \frac{1}{2} \sum_{v \in V(G)} d_D(v).
\]
This index has recently received some attention in the chemical literature [9], because $dd(G)$ certainly carries some interesting structural information for cyclic compounds.

We introduce distance polynomial based on detour distance of a connected graph $G$ defined by
\[
D(G;x) = \sum_{(u,v)} x^{D(u,v)},
\]
where the summation is taken over all unordered pairs $u,v$ of distinct vertices of $G$. Such polynomial of $G$ will be called the detour polynomial (or detour distance polynomial) of $G$. It is clear that
\[
dd(G) = \left. \frac{d}{dx} D(G;x) \right|_{x=1}.
\]
Moreover, one easily notice that
\[
D(G;x) = \sum_{k=1}^{\delta_D} C_D(G,k)x^k,
\]
where $\delta_D = diam_D(G)$, and $C_D(G,k)$ is the number of unordered pairs of distinct vertices $u, v$ such that $D(u,v) = k$. The detour polynomial of a vertex $v$ of $G$ is defined as
\[
D(v;G;x) = \sum_{u \in V(G) \setminus \{v\}} x^{D(v,u)}.
\]
It is clear that
\[
D(G;x) = \frac{1}{2} \sum_{v \in V(G)} D(v,G;x),
\]
and
\[
D(v;G;x) = \sum_{k=1}^{C_D(v)} C_D(v,G;k)x^k,
\]
in which $C_D(v,G;k)$ is the number of vertices $u(\neq v)$ such that $D(u,v) = k$ in $G$. 

In this paper, we find detour polynomials and detour indices for a special class of graphs called ladders, namely $P_n \times K_2$ and Möbius ladder.

2. The Detour Polynomial of a Ladder $L_n$:

A ladder $L_n$ is the graph $P_n \times K_2$, where $P_n$ is the $n$-path, $n \geq 3$. It is clear that $p(L_n) = 2n$, $q(L_n) = 3n - 2$ and $\text{diam}(L_n) = n$. Since $L_n$ is a hamiltonian graph, then $diam_{d}(L_n) = 2n - 1$.

The graph $L_n$ is shown in Fig. 2.1 with the vertices labeled $u_1$, $v_1$, $u_2$, $v_2$, ..., $u_n$, $v_n$.

![Fig. 2.1 The ladder $L_n$, $n \geq 4$](image)

The following theorem determines the detour polynomial for $L_n$, $n \geq 4$.

**Theorem 2.1:**

For $n \geq 4$, we have

$$D(L_n; x) = (n^2 - n + 2)x^{2n-1} + (n^2 - 3n + 4)x^{2n-2} + 2(x^2 + x + 1)\sum_{j=2}^{\left\lfloor \frac{n}{2} \right\rfloor} x^{2n-2j} + \begin{cases} 0, & \text{for even } n, \\ 2x^{n+1} + x^n, & \text{for odd } n. \end{cases} \quad \text{...(2.1)}$$

**Proof:** (I)

First assume $n$ is even. From Fig. 2.1, we find

$$D(u_i, u_j) = \begin{cases} 2n - 1, & \text{for even } j \geq 2, \\ 2n - 2, & \text{for odd } j \geq 3. \end{cases}$$

Also

$$D(u_i, v_j) = \begin{cases} 2n - 1, & \text{for odd } j \geq 1, \\ 2n - 2, & \text{for even } j \geq 2. \end{cases}$$

Therefore, by the symmetry of $L_n$, we obtain

$$D(w, L_n; x) = nx^{2n-1} + (n-1)x^{2n-2}, \ w \in \{u_1, v_1, u_n, v_n\}. \quad \text{...(2.2)}$$

Now, for $i = 2, 3, ..., \frac{n}{2}$; $i \neq j$ and $j \in \{1, 2, ..., n\}$, we have

$$D(u_i, u_j) = \begin{cases} 2n - 1, & \text{if } |j-i| \text{ is odd} \\ 2n - 2, & \text{if } |j-i| \text{ is even} \end{cases}$$

Also, for $i = 2, 3, ..., \frac{n}{2}$ and $j \in \{1, 2, ..., n\} - \{i, i-1, i+1\}$ we have
\[ D(u_j, v_j) = \begin{cases} 2n - 1, & \text{if } |j - i| \text{ is even}, \\ 2n - 2, & \text{if } |j - i| \text{ is odd}. \end{cases} \]

Finally, for \( i = 2, 3, \ldots, \frac{n}{2} \) and \( j = i + 1 \) or \( i - 1 \), we have
\[ D(u_j, v_j) = 2n - \begin{cases} 2i, & \text{if } j = i + 1, \\ 2i - 1, & \text{if } j = i, \\ 2i - 2, & \text{if } j = i - 1. \end{cases} \]

Therefore, for \( i = 2, 3, \ldots, \frac{n}{2} \), we have
\[
D(u_i, L_n; x) = (n - 1)x^{2n-1} + (n - 3)x^{2n-2} + x^{2n-2i} + x^{2n-2i+1} + x^{2n-2i+2}. \quad \text{(2.3)}
\]

It is clear from the Fig. 2.1, that (2.3) holds for \( v_i, u_{n+1-i} \) and \( v_{n+1-i} \), where \( i = 2, 3, \ldots, \frac{n}{2} \). Thus, for even \( n \geq 4 \), we have from (2.2) and (2.3)
\[
D(L_n; x) = \frac{1}{2} \sum_{w \in V(L_n)} D(w, L_n; x)
\]
\[
= \frac{1}{2} \left\{ 4(nx^{2n-1} + (n-1)x^{2n-2}) + 4 \sum_{i=2}^{n} \left[ (n-1)x^{2n-1} + (n-3)x^{2n-2} + x^{2n-2i} + x^{2n-2i+1} + x^{2n-2i+2} \right] \right\}
\]
\[
= (n^2 - n + 2)x^{2n-1} + (n^2 - 3n + 4)x^{2n-2} + 2(x^2 + x + 1) \sum_{i=2}^{n} x^{2n-2i}, \text{ for even } n. \quad \text{(2.4)}
\]

(II) If \( n \) is odd, then using the steps used in proving even case, we get (2.2), and (2.3) for \( i = 2 \) to \( i = \frac{n-1}{2} \). Then, we add \( 2D(u_{\frac{n+1}{2}}, L_n; x) \) inside the brackets \( \{} \), where
\[
D(u_{\frac{n+1}{2}}, L_n; x) = (n - 1)x^{2n-1} + (n - 3)x^{2n-2} + 2x^{n+1} + x^n.
\]

This completes the proof of the theorem. \( \blacksquare \)

For \( L_2 \) and \( L_3 \), we obtain by direct calculation:
\[
D(L_2; x) = 4x^3 + 2x^2,
\]
\[
D(L_3; x) = 8x^5 + 6x^4 + x^3.
\]

**Corollary 2.2:**

For \( n \geq 2 \), we have
\[
dd(L_n) = \begin{cases} 4n^3 - \frac{13}{2} n^2 + 7n - 4, & \text{for even } n, \\ 4n^3 - \frac{13}{2} n^2 + 7n - \frac{7}{2}, & \text{for odd } n. \end{cases} \quad \text{(2.5)}
\]

**Proof:**

Taking the derivative of \( D(L_n; x) \) with respect to \( x \) at \( x = 1 \), we get
\[
\begin{align*}
\frac{d}{dx} D(L_n) &= (n^2 - n + 2)(2n - 1) + (n^2 - 3n + 4)(2n - 2) + 6 \sum_{i=2}^{n} (1)
\end{align*}
\]

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\[ +6 \sum_{i=2}^{n/2} (2n-2i) + \begin{cases} 0, & \text{for even } n, \\ 3n+2, & \text{for odd } n. \end{cases} \]

\[ = 4n^3 - 11n^2 + 19n - 10 + 6(n+1) \left( \frac{n}{2} \right) - 12 \sum_{i=2}^{n/2} i + \begin{cases} 0, & \text{for even } n, \\ 3n+2, & \text{for odd } n. \end{cases} \]

\[ = 4n^3 - 11n^2 + 19n - 10 + \begin{cases} 3(n+1)(n-2) - 6 \left( \frac{n}{2} + 2 \right) \left( \frac{n}{2} - 1 \right), & \text{for even } n, \\ 3(n+1)(n-3) - \frac{3}{2} (n^2 - 9) + 3n + 2, & \text{for odd } n. \end{cases} \]

Simplifying the expression we get (2.5).  

**Remark:**

We notice that the polynomial \( D(L_n; x) \) is of degree \( 2n-1 \), and has \( n \) zeros, with nonzero coefficients \( a_i \) of the terms \( a_i x^i, i = 2n-1, ..., n \).

3. The Detour Polynomial of a Möbius Ladder:

A Möbius ladder, denoted \( ML_n \) is a ladder \( L_n \) with the two edges \( u_i v_i \) and \( v_i u_{i+1} \) as shown in Fig.3.1. It is clear that \( p(ML_n) = 2n, q(ML_n) = 3n, \ diam(ML_n) = \left\lfloor \frac{n}{2} \right\rfloor \) and \( diam_D(ML_n) = 2n - 1 \).

The graph \( ML_n \) is a cubic hamiltonian graph and it can be redrawn as shown in Fig.3.2 from which we see that all its vertices have the same detour polynomial. Thus \( D(ML_n; x) = nD(u_1, ML_n; x). \)  

\[ \text{Fig.3.1 A Möbius ladder } ML_n, \ n \geq 5 \]

\[ \text{Fig.3.2 } ML_n, \ n \geq 5 \]
The detour polynomial of the M"{o}bius ladder is obtained in the next theorem.

**Theorem 3.1:**

For $n \geq 2$,

$$D(ML_n;x) = \begin{cases} n(2n-1)x^{2n-1}, & \text{for even } n, \\ n^2x^{2n-1}+n(n-1)x^{2n-2}, & \text{for odd } n. \end{cases} \quad \ldots(3.2)$$

**Proof:**

Assume that $n \geq 5$. We shall consider two cases for $n$.

(I) $n$ is even.

If $i$ is even, $i \geq 2$, then there is a hamiltonian $u_i - u_i$ path in $ML_n$, namely:

$$u_1, v_1, u_2, u_3, v_3, \ldots, u_{i-1}, v_{i-1}, v_i, v_{i+1}, \ldots, v_n, u_n, u_{n-1}, \ldots, u_{i+1}, u_i.$$ (See Fig.3.1).

Thus

$$D(u_i, u_i) = 2n-1, \quad i = 2, 4, \ldots, n. \quad \ldots(3.3)$$

If $i$ is odd, $i \geq 3$, there is also a hamiltonian $u_i - u_i$ path in $ML_n$, namely:

$$u_1, u_2, \ldots, u_{i-1}, v_{i-1}, v_{i-2} \ldots, v_2, v_1, u_n, v_n, v_{n-1}, u_{n-1}, u_{n-2}, v_{n-2}, v_{n-3}, \ldots, v_i, u_i.$$ Therefore

$$D(u_i, u_i) = 2n-1, \quad i = 3, 5, \ldots, n-1. \quad \ldots(3.4)$$

Now, we determine $D(u_i, v_i)$. If $i$ is even, $i \geq 2$, then there is a hamiltonian $u_i - v_i$ path, namely:

$$u_1, u_2, \ldots, u_{i-1}, v_{i-1}, v_{i-2} \ldots, v_2, v_1, u_n, v_n, v_{n-1}, u_{n-1}, u_{n-2}, v_{n-2}, v_{n-3}, \ldots, u_i, v_i.$$ Thus

$$D(u_i, v_i) = 2n-1, \quad i = 2, 4, \ldots, n. \quad \ldots(3.5)$$

If $i$ is odd, $i \geq 1$, then there is a hamiltonian $u_i - v_i$ path, namely (this is for $i \geq 3$, for $i = 1$, it is clear):

$$u_1, v_1, v_2, u_2, \ldots, u_{i-1}, \ldots, u_{i+1}, u_n, v_n, v_{n-1}, v_{n-2}, v_{n-3}, \ldots, v_i, v_i.$$ Thus

$$D(u_i, v_i) = 2n-1, \quad i = 1, 3, 5, \ldots, n-1. \quad \ldots(3.6)$$

Hence, for every vertex $w \neq u_i$ of $ML_n$, we have $D(u_i, w) = 2n-1$. Thus, from (3.1), we obtain (3.2) for even $n$.

(II) $n$ is odd.

Suppose that $i$ is even, then it is clear from Fig.3.1, that

$$u_1, v_1, v_2, u_2, u_3, v_3, \ldots, u_{i-1}, v_{i-1}, v_i, v_{i+1}, \ldots, v_n, u_n, u_{n-1}, u_{n-2}, \ldots, u_i,$$

is a hamiltonian $u_i - u_i$ path for even $i$. Thus

$$D(u_i, u_i) = 2n-1, \quad i = 2, 4, \ldots, n-1. \quad \ldots(3.7)$$

If $i$ is odd, then

$$u_1, v_1, v_2, u_2, u_3, v_3, \ldots, u_{i-2}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, \ldots, v_n, u_n, u_{n-1}, u_{n-2}, \ldots, u_i,$$

which does not contain $u_{i-1}$ is a $u_i - u_i$ detour of length $2n-2$, for odd $i$. Thus

$$D(u_i, u_i) = 2n-2, \quad i = 1, 3, 5, \ldots, n. \quad \ldots(3.8)$$

To find $D(u_i, v_i)$, first assume $i$ is even, then

$$u_1, v_1, v_2, u_2, u_3, v_3, \ldots, v_{i-2}, u_{i-2}, u_{i-1}, \ldots, u_n, v_n, v_{n-1}, v_{n-2}, \ldots, v_i,$$

which does not contain $v_{i-1}$ is a $u_i - v_i$ detour of length $2n-2$ for even $i$. Thus
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\[ D(u_i, v_i) = 2n - 2, \text{ for } i = 2, 4, \ldots, n - 1. \] 
\[ (3.9) \]

Now, let \( i \) be odd, then there is a hamiltonian \( u_i - v_i \) path

\[ u_1, v_1, u_2, v_2, u_3, v_3, \ldots, v_{i-1}, u_{i-1}, u_i, u_{i+1}, \ldots, u_n, v_n, v_{n-1}, v_{n-2}, \ldots, v_i. \]

Thus

\[ D(u_i, v_i) = 2n - 1, \text{ for } i = 1, 3, 5, \ldots, n. \] 
\[ (3.10) \]

From (3.7) and (3.10) we get \( n \) pairs of vertices \((u_i, u_j)\) and \((u_i, v_j)\) of detour distance \(2n - 1\); and from (3.8) and (3.9), we get \((n-1)\) pairs of vertices \((u_i, u_j)\) and \((u_i, v_j)\) of detour distance \(2n - 2\). Thus, from (3.1) we obtain (3.2) for odd \( n \).

By direct calculation one may easily obtain:

\[ \begin{align*}
D(ML_2; x) &= 6x^3, \\
D(ML_3; x) &= 9x^5 + 6x^4, \\
D(ML_4; x) &= 28x^7, \\
D(ML_5; x) &= 25x^9 + 20x^8,
\end{align*} \]

which are the same results obtained from (3.2). Thus, the Theorem 3.1 holds for all values of \( n \geq 2 \).

From Theorem 3.1 and using (1.9) we get \( dd(ML_n) \) as given in the next corollary:

**Corollary 3.2:**

For \( n \geq 2 \), the detour index of \( ML_n \) is

\[ dd(ML_n) = \begin{cases} 
 n(2n - 1)^2, & \text{for even } n, \\
 4n^3 - 5n^2 + 2n, & \text{for odd } n.
\end{cases} \]

The following corollary is a useful graph theoretical result.

**Corollary 3.3:**

The Möbius ladder \( ML_n \) is hamiltonian-connected if and only if \( n \) is even.

A connected graph \( G \) of order \( p \) is called **saturated** (with respect to detour distance) [9] if \( dd(G) = \frac{1}{2} \ p(p-1)^2 \); that is if and only if \( G \) is a hamiltonian-connected graph. Thus \( ML_n \) is saturated if \( n \) is even.

The **density** [9] of a \((p, q)\) connected graph \( G \) is defined as \( \text{den}(G) = \frac{q}{p} \). One may show that the density of every saturated graph \( G \) is not less than \( \frac{3}{2} \). Thus, from corollary 3.3, \( ML_n \) for even \( n \), is saturated with minimum density \( \frac{3}{2} \).
REFERENCES

[1] F. Buckley, and F. Harary (1990), Distance in Graphs, Addison-Wesley, Redwood City, California.


