Traveling Wave Solutions using Tanh Method for solving Kawahara and Modified Kawahara Equations

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Abstract: *This paper implemented very reliable technique which is called the Tanh method for solving evolution equations. The Tanh method has been successfully tested on two important Kawahara and modified Kawahara equations. The calculations demonstrate the effectiveness and convenience of Tanh method for solving nonlinearPDEs.*

Keywords: ***Tanh method, Nonlinear system PDEs, Exact Solutions, Kawahara and modified Kawahara equations.***

1. Introduction

Nonlinear coupled partial differential equations are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves and chemical physics. The nonlinear wave phenomena observed in the above mentioned scientific fields, are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions, for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution. Several direct methods for obtaining the explicit travelling solitary wave solutions to nonlinear evolution equations have been recently proposed, such as the homogeneous balance method [1], the tanh-function method [2], the sine–cosine method [3], the extended tanh-function method [4], the modified tanh-function method [5] and the Jacobi elliptic function expansion method [6], and so on. The key idea of these generalized methods are to use the solutions of an auxiliary ordinary differential equation to replace tanh(ζ) in the tanh-function method.

This paper outlines the implementation of very efficient and reliable technique which is called Tanh method for solving Kawahara and modified Kawahara equations which are very important in applied sciences. The Tanh method is a powerful technique to symbolically compute traveling wave solutions of one-dimensional nonlinear wave and evolution equations. In particular, the method is well suited for problems where dispersion, convection, and reaction diffusion phenomena play an important role [7].

2. Outline of the Tanh Method

The tanh method will be introduced as presented by Malfliet [8] and by Wazwaz [9–11]. The tanh method is based on a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function to solve nonlinear Differential Equations.

The tanh method is developed by Malfliet [8]. The method is applied to find out exact solutions of nonlinear differential equations:

$$P(u, u_t, u_x, u_{xx}, ...) = 0 \tag{1}$$

where P is a polynomial of the variable u and its derivatives. Consider the transformation variable $\xi = k(x + \lambda t)$, so that $u(x,t) = U(\xi)$
then can use the following changes by chain rule:

$$\frac{\partial}{\partial t} = k\lambda \frac{d}{d\xi}, \qquad \frac{\partial}{\partial x} = k \frac{d}{d\xi},$$

$$\frac{\partial^2}{\partial x^2} = k^2 \frac{d^2}{d\xi^2}, \qquad \frac{\partial^3}{\partial x^3} = k^3 \frac{d^3}{d\xi^3},$$

and so on, then Eq. (1) becomes an ordinary differential equation

$$Q(U, U^{/}, U^{//}, ...) = 0 \tag{2}$$

With Q being another polynomials form of there argument, which will be called the reduced ordinary differential equations of Eq.(1). Integrating Eq.(2) as long as all terms contain derivatives, the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well. Now finding the traveling wave solutions to Eq.(1) is equivalent to obtaining the solution to the reduced ordinary differential Eq. (2). For the tanh method, we introduce the new independent variable

$$Y(x,t) = \tanh(\xi) \tag{3}$$

that leads to the change of variables:

$$\frac{d}{d\xi} = (1-Y^2)\frac{d}{dY}$$

$$\frac{d^2}{d\xi^2} = -2Y(1-Y^2)\frac{d}{dY} + (1-Y^2)^2\frac{d^2}{dY^2}$$

$$\frac{d^3}{d\xi^3} = 2(1-Y^2)(3Y^2-1)\frac{d}{dY} - 6Y(1-Y^2)^2\frac{d^2}{dY^2} + (1-Y^2)^3\frac{d^3}{dY^3} \quad (4)$$

$$\frac{d^4}{d\xi^4} = 8Y(2-3Y^2)(1-Y^2)\frac{d}{dY} - 4(2-9Y^2)(1-Y^2)^2\frac{d^2}{dY^2} - 12Y(1-Y^2)^3\frac{d^3}{dY^3} + (1-Y^2)^4\frac{d^4}{dY^4}$$

The next crucial step is that the solution is expressed in the form:

$$u(x,t) = U(\xi) = \sum_{i=0}^{m} a_i Y^i \quad (5)$$

where the parameters m, n can be found by balancing the highest-order linear term with the nonlinear terms in Eq.(2), and $k, \lambda, a_0, a_1, ..., a_m$ are to be determined. Substituting Eq.(5) into Eq.(2) will yield a set of algebraic equations for $k, \lambda, a_0, a_1, ..., a_m$ because all coefficients of Y have to vanish. From these relations, $k, \lambda, a_0, a_1, ..., a_m$ can be obtained. Having determined these parameters, knowing that m , n are positive integers in most cases, and using Eq.(5) we obtain analytic solutions $u(x,t)$ in a closed form [12]. The tanh method seems to be powerful tool in dealing with nonlinear physical models.

3. Numerical Applications:

The tanh method is generalized on two examples including Kawahara and modified Kawahara equations.

Example 1. Consider the following Kawahara equation [13]:

$$u_t + \alpha.uu_x + \beta.u_{xxx} - \gamma.u_{xxxxx} = 0 \qquad (6)$$

Where α, β, γ are nonzero constants. Kawahara equation (6) occurs in the theory of magneto-acoustic waves in a plasmas [13] and in the theory of shallow water waves with surface tension [14]. For $\alpha = \beta = \gamma = 1$ Eq.(6) becomes:

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0 \qquad (7)$$

Making the transformation $u(x,t) = u(\xi)$, and $\xi = x - \lambda t$ then integrating once with respect to ξ with zero constant, Eq.(7) becomes:

$$-\lambda U + \frac{1}{2}U^2 + U^{//} - U^{(4)} = 0 \qquad (8)$$

we postulate the following tanh series, and the transformation given in Eq.(4), then Eq.(8) reduces to:

$$-\lambda U+\frac{1}{2}U^2-2Y(1-Y^2)\frac{dU}{dY}+(1-Y^2)^2\frac{d^2U}{dY^2}$$
$$-8Y(2-3Y^2)(1-Y^2)\frac{dU}{dY}$$
$$+4(2-9Y^2)(1-Y^2)^2\frac{d^2U}{dY^2}+12Y(1-Y^2)^3\frac{d^3U}{dY^3} \qquad (9)$$
$$-(1-Y^2)^4\frac{d^4U}{dY^4}=0$$

Now, to determine the parameter m we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq. (9) we balance $U^{(4)}$ with U^2, to obtain 8+ m – 4 = 2m, then m = 4.

The tanh method admits the use of the finite expansion for :

$$u(x,t)=U(Y)=a_0+a_1Y+a_2Y^2+a_3Y^3+a_4Y^4, \quad a_4\neq 0 \qquad (10)$$

Substituting $U^{/},U^{//},U^{///},U^{(4)}$ in Eq. (9), then equating the coefficient of Y^i, i= 0, 1, 2, 3,….,8 leads to the following nonlinear system of algebraic equations:

$$Y^0: \quad -\lambda a_0+\frac{1}{2}a_0^2+18a_2-24a_4=0$$
$$Y^1: -\lambda a_1+a_0a_1-18a_1+126a_3=0$$
$$Y^2: -\lambda a_2+a_0a_2+\frac{1}{2}a_1^2+108a_4+144a_2=0 \qquad (11)$$

$$Y^3: \quad -\lambda a_3+a_0a_3+a_1a_2+42a_1-594a_3+96a_4=0$$
$$Y^4: -\lambda a_4+a_0a_4+a_1a_3+\frac{1}{2}a_2^2-42a_2-1248a_4=0$$
$$Y^5: a_1a_4+a_2a_3+828a_3-172a_4-24a_1=0$$

$$Y^6 : a_2 a_4 + \frac{1}{2} a_3^2 - 72 a_2 - 608 a_4 = 0$$

$$Y^7 : a_3 a_4 - 360 a_3 + 96 a_4 = 0$$

$$Y^8 : \frac{1}{2} a_4^2 - 408 a_4 = 0$$

Solving the nonlinear systems of equations (11) we can get the following cases:

Case 1. $a_0 = 2252$, $\lambda = 1124$, $a_4 = 816$, $a_3 = -171$, $a_2 = 647$, $a_1 = 495$

Then:

$$u_1(x, t) = 2252 + 495 \tanh(x - 1124\,t) + 647 \tanh^2(x - 1124\,t)$$
$$-171 \tanh^3(x - 1124\,t) + 816 \tanh^4(x - 1124\,t) \qquad (12)$$

Case 2. $a_0 = 0$, $\lambda = 3015$, $a_4 = 816$, $a_3 = -171$, $a_2 = 647$, $a_1 = 495$

Then:

$$u_2(x, t) = 495 \tanh(x - 3015\,t) + 647 \tanh^2(x - 3015\,t)$$
$$-171 \tanh^3(x - 3015\,t) + 816 \tanh^4(x - 3015\,t) \qquad (13)$$

Case 3. $a_0 = 0$, $\lambda = 460$, $a_4 = 816$, $a_3 = -171$, $a_2 = 647$, $a_1 = 495$

Then:

$$u_3(x,t) = 495\tanh(x-460t) + 647\tanh^2(x-460t) - 171\tanh^3(x-460t) + 816\tanh^4(x-460t) \quad (14)$$

Case 4. $a_0 = 61+65i$, $\lambda = 65i$, $a_4 = 816$, $a_3 = -171$, $a_2 = 647$, $a_1 = 495$

Then:

$$u_4(x,t) = 61+65i + 495\tanh(x-65it) + 647\tanh^2(x-65it) - 171\tanh^3(x-65it) + 816\tanh^4(x-65it) \quad (15)$$

The solitary wave and behavior of the solution $u_1(x,t)$ is shown in Figure (1).

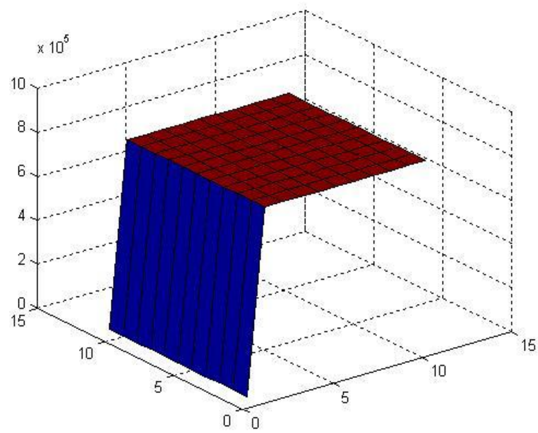

Figure (1). The solitary wave and behavior of the solution $u_1(x,t)$. For $0 \le t \le 1$ and $0 \le x \le 1$

Example 2. we consider the modified Kawahara equation

$$u_t + u_x + u^2 u_x + p u_{xxx} - q u_{xxxxx} = 0 \quad (16)$$

where *p*, *q* are nonzero real constants. This equation also arises in the theory of shallow water waves [14]. Making the transformation $u(x,t)=u(\xi)$, and $\xi = x+\lambda t$ then integrating once with respect to ξ with zero constant, Eq.(16) becomes:

$$(\lambda+1)u+\frac{1}{3}u^3+pu^{//}-qu^{(4)}=0 \qquad (17)$$

we postulate the following tanh series, and the transformation given in Eq.(4), then Eq.(17) reduces to:

$$(\lambda+1)U+\frac{1}{3}U^3+p[-2Y(1-Y^2)\frac{dU}{dY}+(1-Y^2)^2\frac{d^2U}{dY^2}]$$
$$-q[8Y(2-3Y^2)(1-Y^2)\frac{dU}{dY}-4(2-9Y^2)(1-Y^2)^2\frac{d^2U}{dY^2}$$
$$-12Y(1-Y^2)^3\frac{d^3U}{dY^3}+(1-Y^2)^4\frac{d^4U}{dY^4}]=0 \qquad (18)$$

Now, to determine the parameter m we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq.(18) we balance $U^{(4)}$ with U^3, to obtain 8+ m – 4 = 3m, then m = 2. The tanh method admits the use of the finite expansion for :

$$u(x,t)=U(Y)=a_0+a_1Y+a_2Y^2, \quad a_2\neq 0 \qquad (19)$$

Substituting $U^{/},U^{//},U^{///},U^{(4)}$ in Eq. (18) we get:

$$(\lambda+1)(a_0+a_1Y+a_2Y^2)+(\frac{1}{3}a_0^3+a_0^2a_1Y+(a_0^2a_2+a_1^2a_0)Y^2$$
$$+(\frac{1}{3}a_1^3+2a_0a_1a_2)Y^3+(a_2^2a_0+a_1^2a_2)Y^4+a_1a_2^2Y^5+\frac{1}{3}a_2^3Y^6)$$
$$+2p[a_2-Ya_1-4a_2Y^2+a_1Y^3+3a_2Y^4]$$
$$+q[16a_2-8a_1Y-136a_2Y^2+40a_1Y^3+240a_2Y^4$$
$$-24a_1Y^5-120a_2Y^6]=0$$

(20)

By equating the coefficient of Y^i, i= 0, 1, 2, 3,….,6 in Eq.(20) leads to the following nonlinear system of algebraic equations:

$$\overset{0}{Y}:(\lambda+1)a_0+\frac{1}{3}a_0^3+2pa_2+q16a_2=0$$
$$\overset{1}{Y}:(\lambda+1)a_1+a_0^2a_1-2pa_1-8qa_1=0$$
$$\overset{2}{Y}:(\lambda+1)a_2+a_0^2a_2+a_1^2a_0-8pa_2-136qa_2=0$$
$$\overset{3}{Y}:\frac{1}{3}a_1^3+2a_0a_1a_2+2pa_1+40qa_1=0 \qquad (21)$$
$$\overset{4}{Y}:a_2^2a_0+a_1^2a_2+6pa_2+240qa_2=0$$
$$\overset{5}{Y}:a_1a_2^2-24qa_1=0$$
$$\overset{6}{Y}:\frac{1}{3}a_2^3-120qa_2=0$$

Solving the nonlinear systems of equations (21) we can get the following cases:

Case 1. $a_0 = 0$, $a_1 = 8i\sqrt{3q}$, $a_2 = 6\sqrt{10q}$, $\lambda = 72q - 1$ (22)

Then:

$$u_1(x, t) = \sqrt{q}\{8\sqrt{3}\,\tan i[x + (72q - 1)t] + 6\sqrt{10}\,\tan h^2[x + (72q - 1)t]\} \quad (23)$$

Case 2. $a_0 = 0$, $a_1 = 8i\sqrt{3q}$, $a_2 = 2\sqrt{6q}$, $\lambda = 72q - 1$

Then:

$$u_2(x, t) = \sqrt{q}\{8\sqrt{3}\,\tan i[x + (72q - 1)t] + 2\sqrt{6}\,\tan h^2[x + (72q - 1)t]\} \quad (24)$$

Case 3. $a_0 = 0, a_1 = 6i\sqrt{2q}$, $a_2 = 6\sqrt{10q}$, $\lambda = 72q - 1$

Then:

$$u_3(x, t) = \sqrt{q}\{6\sqrt{2}\,\tan i[x + (72q - 1)t] + 6\sqrt{10}\,\tan h^2[x + (72q - 1)t]\} \quad (25)$$

Case 4. $a_0 = 0, a_1 = 6i\sqrt{2q}$, $a_2 = 2\sqrt{6q}$, $\lambda = 72q - 1$

Then:

$$u_4(x, t) = \sqrt{q}\{6\sqrt{2}\,\tan i[x + (72q - 1)t] + 2\sqrt{6}\,\tan h^2[x + (72q - 1)t]\} \quad (26)$$

Case 5. $a_0 = 0$, $a_1 = 8i\sqrt{3q}$, $a_2 = 6\sqrt{10q}$, $\lambda = -(8q + 1)$

Then:

$$u_5(x, t) = \sqrt{q}\{8\sqrt{3}\,\tan i[x - (8q + 1)t] + 6\sqrt{10}\,\tan h^2[x - (8q + 1)t]\} \quad (27)$$

Case 6. $a_0 = 0$, $a_1 = 8i\sqrt{3q}$, $a_2 = 2\sqrt{6q}$, $\lambda = -(8q + 1)$

Then:

$$u_6(x,t)=\sqrt{q}\{8\sqrt{3}\tan i\ [x-(8q+1)t]+2\sqrt{6}\tan h^2\ [x-(8q+1)t]\} \quad (28)$$

Case 7. $a_0=0,\ a_1=6i\sqrt{2q},\ a_2=6\sqrt{10q},\ \lambda=-(8q+1)$
Then:

$$u_7(x,t)=\sqrt{q}\{6\sqrt{2}\tan i[x-(8q+1)t]+6\sqrt{10}\tan h^2\ [x-(8q+1)t]\} \quad (29)$$

Case 8. $a_0=0,\ a_1=6i\sqrt{2q},\ a_2=2\sqrt{6q},\ \lambda=-(8q+1)$
Then:

$$u_8(x,t)=\sqrt{q}\{6\sqrt{2}\tan i[x-(8q+1)t]+2\sqrt{6}\tan h^2\ [x-(8q+1)t]\} \quad (30)$$

4. Conclusion:

The tanh method was employed for analytic treatment of nonlinear partial differential equations. The tanh method requires transformation formulas. Traveling wave solutions, kinks solutions were derived. Solutions of two examples by the Tanh Method demonstrate the effectiveness and convenience in solving Nonlinear PDEs compared with other methods.

References

[1] Wang ML. Phys Lett A (1995); Vol.199, p. 169.

[2] Parkes EJ, Duffy BR. Comp Phys Comm (1996), Vol.98, p.288.

[3] Yan C. Phys Lett A (1996) , Vol.224, p.74.

[4] Fan EG. Phys Lett A (2000),Vol.277, p.212.

[5] Elwakil SA, El-labany SK, Zaharan MA, Sabry R. Phys Lett A (2002) Vol.299, p.179.

[6] Parkes EJ, Duffy BR, Abbott PC. Phys Lett A (2002) Vol.295, p.280.

[7] W. Malfliet, The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations, J. Comp. Appl. Math, (2004), Vol.164, pp.(529-541).

[8] W. Malfliet, Solitary wave solutions of nonlinear wave equations, Am. J. Phys. (1992),Vol.60, pp.(650–654).

[9] Abdul-Majid Wazwaz, Exact and explicit travelling wave solutions for the nonlinear Drinfeld Sokolov system, Comm. Nonlinear Sci. Num. Siml., (2006), Vol.11 No.3, pp.(311–325).

[10] Abdul-Majid Wazwaz, Two reliable methods for solving variants of the KdV equation with compact and non compact structures, Chaos Solitons Fract. (2006),Vol.28, No.2, pp.(454–462).

[11] Abdul-Majid Wazwaz, The tanh method for compact and non compact solutions for variants of the KdV–Burger equations, Phys. D: Nonlinear Phenomena, (2006), Vol. 213,No.2, pp.(147–151).

[12] Sirendaoreji, New exact travelling wave solutions for the Kawahara and modified Kawahara equations, Chaos, Solitons and Fractals, (2004), Vol.19, 147–150

[13] Kawahara T. J Phys Soc Jpn, (1972),Vol.33, p.260.

[14] Hunter JK, Scheule J. Physica D, (1988), Vol.32, p.253.

حلول معادلات (Kawahra) باستخدام طريقة الظل الزائدي

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المستخلص

تم في هذا البحث تطبيق طريقة مقترحة وهي (الظل الزائدي) لحل المعادلات الجزئية التفاضلية. وتم تطبيق الطريقة المقترحة لحل معادلات مهمة تطبيقية وهي معادلات(Kawahara , Kawahara المعدلة) بينت الحسابات سهولة أسلوب تطبيق الطريقة المقترحة وموثوقيتها وكفاءتها في حل المعادلات التفاضلية الجزئية والتي تشجع لحل المعادلات التفاضلية الجزئية في المستقبل.