On S*-Convergence Nets And Filters

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Abstract: This paper is devoted to introduce and study many topological properties of s*-convergence of nets and s*-convergence of filters by using the concept of s*-open sets, also some properties of s*-cluster points of nets and filters has been studied.

Key words: s*-open, s*-closed, s*-convergent, s*-cluster , s*-limit point and s*-irresolute
1. Introduction

The concept of $s^*$-closed set was first introduced by Al-Meklafi, S. [1], by using the concept of semi-open set. Recall that a subset $A$ of a topological space $(X, \tau)$ is called semi-open (briefly $s$-open) set if there exists an open subset $U$ of $X$ such that $U \subseteq A \subseteq \text{cl}(U)$. The complement of a semi-open set is defined to be semi-closed (briefly $s$-closed) [2]. Also, a subset $A$ of a topological space $(X, \tau)$ is called $s^*$-closed if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$ [1]. The complement of an $s^*$-closed set is defined to be $s^*$-open. The family of all $s^*$-open (resp. $s^*$-closed) subsets of $(X, \tau)$ is denoted by $S^O(X, \tau)$ (resp. $S^S(X, \tau)$) [1].

Throughout this paper $(X, \tau)$ and $(Y, \tau^*)$ (or simply $X$ and $Y$) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

2. $S^*$-Convergence Of Nets

2.1. Definition:
A subset $A$ of a topological space $X$ is called an $s^*$-neighborhood of a point $x$ in $X$ if there exists an $s^*$-open set $U$ in $X$ such that $x \in U \subseteq A$. The family of all $s^*$-neighborhoods of a point $x \in X$ is denoted by $N_{s^*}(x)$.

2.2. Remark:
Since every open set is an $s^*$-open, then every neighborhood of $x$ is an $s^*$-neighborhood of $x$, but the converse is not true in general. Consider the following example :-
Example:
Let \( X \) any infinite set with indiscrete topology and \( x \in X \), then \( \{ x \} \) is an \( s^* \)-neighborhood of \( x \), since \( x \in \{ x \} \subseteq \{ x \} \), where \( \{ x \} \) is an \( s^* \)-open set in \( X \), while \( \{ x \} \) is not a neighborhood of \( x \).

2.3. Theorem:
A function \( f : X \to Y \) from a topological space \( X \) to a topological space \( Y \) is \( s^* \)-irresolute iff for each \( x \in X \) and each \( s^* \)-neighborhood \( V \) of \( f(x) \) in \( Y \), there is an \( s^* \)-neighborhood \( U \) of \( x \) in \( X \) such that \( f(U) \subseteq V \).

Proof: \( \Rightarrow \)
Let \( f : X \to Y \) be an \( s^* \)-irresolute function and \( V \) be an \( s^* \)-neighborhood of \( f(x) \) in \( Y \). To prove that, there is an \( s^* \)-neighborhood \( U \) of \( x \) in \( X \) such that \( f(U) \subseteq V \). Since \( f \) is an \( s^* \)-irresolute then, \( f^{-1}(V) \) is an \( s^* \)-neighborhood of \( x \) in \( X \).
Let \( U = f^{-1}(V) \Rightarrow f(U) = f(f^{-1}(V)) \subseteq V \Rightarrow f(U) \subseteq V \).

Conversely,
To prove that \( f : X \to Y \) is \( s^* \)-irresolute. Let \( V \) be an \( s^* \)-open set in \( Y \). To prove that \( f^{-1}(V) \) is an \( s^* \)-open in \( X \). Let \( x \in f^{-1}(V) \Rightarrow f(x) \in V \Rightarrow V \) is an \( s^* \)-neighborhood of \( f(x) \). By hypothesis there is an \( s^* \)-neighborhood \( U_x \) of \( x \) such that \( f(U_x) \subseteq V \).

\[ \Rightarrow U_x \subseteq f^{-1}(V), \forall x \in f^{-1}(V) \Rightarrow \exists \text{ an } s^*\text{-open set } W_x \text{ of } x \text{ such that } W_x \subseteq U_x \subseteq f^{-1}(V), \forall x \in f^{-1}(V) \Rightarrow \bigcup_{x \in f^{-1}(V)} W_x \subseteq f^{-1}(V). \]

Since \( f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \{ x \} \subseteq \bigcup_{x \in f^{-1}(V)} W_x \Rightarrow f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} W_x \).

\[ \Rightarrow f^{-1}(V) \text{ is an } s^*\text{-open in } Y, \text{ since its a union of } s^*\text{-open sets}. \]
Thus \( f : X \to Y \) is an \( s^* \)-irresolute function.
2.4. Definition:
Let \((x_d)_{d \in D}\) be a net in a topological space \(X\). Then \((x_d)_{d \in D}\) \(s^*\)-converges to \(x \in X\) (written \(x_d \xrightarrow{s^*} x\)) iff for each \(s^*\)-neighborhood \(U\) of \(x\), there is some \(d_0 \in D\) such that \(d \geq d_0\) implies \(x_d \in U\). Thus \(x_d \xrightarrow{s^*} x\) iff each \(s^*\)-neighborhood of \(x\) contains a tail of \((x_d)_{d \in D}\). This is sometimes said \((x_d)_{d \in D}\) \(s^*\)-converges to \(x\) iff it is eventually in every \(s^*\)-neighborhood of \(x\). The point \(x\) is called an \(s^*\)-limit point of \((x_d)_{d \in D}\).

2.5. Definition:
Let \((x_d)_{d \in D}\) be a net in a topological space \(X\). Then \((x_d)_{d \in D}\) is said to have \(x \in X\) as an \(s^*\)-cluster point (written \(x_d \xrightarrow{\mathcal{A}} x\)) iff for each \(s^*\)-neighborhood \(U\) of \(x\) and for each \(d \in D\), there is some \(d_0 \geq d\) such that \(x_{d_0} \in U\). This is sometimes said \((x_d)_{d \in D}\) has \(x\) as an \(s^*\)-cluster point iff \((x_d)_{d \in D}\) is frequently in every \(s^*\)-neighborhood of \(x\).

2.6. Theorem:
Let \(A\) be a subset of a topological space \(X\). Then \(x \in s^*\text{cl}(A)\) if and only if for any \(s^*\)-open set \(U\) containing \(x\), \(A \cap U \neq \emptyset\).

Proof: \(\Rightarrow\)
Let \(x \in s^*\text{cl}(A)\) and suppose that, there is an \(s^*\)-open set \(U\) in \(X\) such that \(x \in U \& A \cap U = \emptyset \Rightarrow A \subseteq U^c\) which is \(s^*\)-closed in \(X\) \(\Rightarrow s^*\text{cl}(A) \subseteq U^c\).
\(\therefore x \in U \Rightarrow x \notin U^c \Rightarrow x \notin s^*\text{cl}(A)\), this is a contradiction.

Conversely,
Suppose that, for any \(s^*\)-open set \(U\) containing \(x\), \(A \cap U \neq \emptyset\).
To prove that \(x \in s^*\text{cl}(A)\), if not \(\Rightarrow x \notin s^*\text{cl}(A)\)
\[ x \in (s^* - cl(A))^c \text{ which is } s^*-\text{open in } X \Rightarrow A \cap (s^* - cl(A))^c \neq \emptyset. \]

This is a contradiction, since \( A \cap (s^* - cl(A))^c = \emptyset \). Thus \( x \in s^* - cl(A) \).

Since every neighborhood is an \( s^* \)-neighborhood, then we have the following theorem:

\subsection*{2.7. Theorem:}
Let \( X \) be a topological space and \( (x_d)_{d \in D} \) be a net in \( X \) and \( x \in X \). Then:

\begin{enumerate}[i)]
  \item If \( x_d \overset{s^*}{\to} x \), then \( x_d \not\approx x \).
  \item If \( x_d \overset{s^*}{\to} x \) \( (x_d \not\approx x) \), then \( x_d \to x \) \( (x_d \not\approx x) \) respectively.
\end{enumerate}

\subsection*{2.8. Remarks:}

1) The converse of (2.7(i)) may not be true in general. To show that we give the following example:

\textbf{Example:} Let \((\mathcal{R}, \mu)\) be the usual topological space where \( \mathcal{R} \) be the set of all real numbers, then the net \( (s_n)_{n \in N} = (n + (-1)^n n)_{n \in N} \) in \( \mathcal{R} \) has \( 0 \) as an \( s^* \)-cluster point but not \( s^* \)-limit point. Since if \( U \) is an \( s^* \)-neighborhood of \( 0 \) in \( \mathcal{R} \), then for each \( n \in N \), either \( n \) is odd or even. If \( n \) is odd, then \( n_0 = n \Rightarrow s_{n_0} = 0 \in U \) and if \( n \) is even, then \( n_0 = n + 1 \Rightarrow s_{n_0} = 0 \in U \), thus \( s_n \not\approx 0 \). But \( s_n \) does not \( s^* \)-converge to \( 0 \), since \( U = (-1,1) \) is an \( s^* \)-neighborhood of \( 0 \) and \( s_n \not\in (-1,1), \forall n \in N \).

2) The converse of (2.7(ii)) may not be true in general. To show that we give the following example:
Example: Let \( (N, I) \) be the indiscrete topological space where \( N \) be the set of all natural numbers and \( (s_n)_{n \in N} = (n)_{n \in N} \) be a net in \( N \). Observe that \( s_n \to 1 (s_n \simeq 1) \). But \( s_n \) does not \( s^*-\)converge to 1 (does not \( s^*-\)cluster to 1), since \( \{1\} \) is an \( s^*-\)neighborhood of 1 and \( s_n \notin \{1\}, \forall n > 1 \).

2.9. Theorem:
Let \( X \) be a topological space and \( A \subseteq X \). If \( x \) is a point of \( X \), then \( x \in s^*-cl(A) \) if and only if there exists a net \( (x_d)_{d \in D} \) in \( A \) such that \( x_d \xrightarrow{s^*} x \).

Proof: \( \Leftarrow \)
Suppose that \( \exists \) a net \( (x_d)_{d \in D} \) in \( A \) such that \( x_d \xrightarrow{s^*} x \). To prove that \( x \in s^*-cl(A) \). Let \( U \in N_{s^*}(x) \), since
\[
x_d \xrightarrow{s^*} x \Rightarrow \exists d_0 \in D \text{ such that } x_d \in U \ \forall d \geq d_0.
\]
But \( x_d \in A \ \forall d \in D. \Rightarrow U \cap A \neq \emptyset \ \forall U \in N_{s^*}(x) \). Hence by (2.6), we get \( x \in s^*-cl(A) \).

Conversely,
Suppose that \( x \in s^*-cl(A) \). To prove that \( \exists \) a net \( (x_d)_{d \in D} \) in \( A \) such that \( x_d \xrightarrow{s^*} x \).
\[
\therefore x \in s^*-cl(A) \text{, then by (2.6), we get } N \cap A \neq \emptyset \ \forall N \in N_{s^*}(x).
\]
\[
\therefore D = N_{s^*}(x) \text{ is a directed set by inclusion .}
\]
\[
\therefore N \cap A \neq \emptyset \ \forall N \in N_{s^*}(x) \Rightarrow \exists x_N \in N \cap A.
\]
Define \( x : N_{s^*}(x) \to A \) by : \( x(N) = x_N \ \forall N \in N_{s^*}(x) \).
\[
\therefore (x_N)_{N \in N_{s^*}(x)} \text{ is a net in } A. \text{ To prove that } x_N \xrightarrow{s^*} x.
\]
Let \( N \in N_{s^*}(x) \) to find \( d_0 \in D \) such that \( x_d \in N \ \forall d \geq d_0 \).
Let \( d_0 = N \Rightarrow \forall d \geq d_0 \Rightarrow d = M \in N_{s^*}(x) \).
i.e. \( M \geq N \iff M \subseteq N \).
\[
\therefore x_d = x(d) = x(M) = x_M \in M \cap A \subseteq M \subseteq N \Rightarrow x_M \in N.
\]
\[ x_d \in N \ \forall \ d \geq d_0. \] Thus \[ x_N \xrightarrow{s^*} x. \]

**2.10. Definition:** [4]

A topological space \( X \) is called an \( s^*-T_2 \)-space if for any two distinct points \( x \) and \( y \) of \( X \), there are two \( s^* \)-open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

**2.11. Theorem:**

A topological space \( X \) is an \( s^*-T_2 \)-space iff every \( s^* \)-convergent net in \( X \) has a unique \( s^* \)-limit point.

**Proof:** \( \Rightarrow \)

Let \( X \) be an \( s^*-T_2 \)-space and \( (x_d)_{d \in D} \) be a net in \( X \) such that \( x_d \xrightarrow{s^*} x \) \& \( x_d \xrightarrow{s^*} y \) \& \( x \neq y \). Since \( X \) is an \( s^*-T_2 \)-space, \( \exists U \in N_{s^*}(x) \) and \( V \in N_{s^*}(y) \) such that \( U \cap V = \emptyset \).

\[ x_d \xrightarrow{s^*} x \Rightarrow \exists d_0 \in D \ \ s.t \ \ x_d \in U \ \ \forall \ d \geq d_0. \]

\[ x_d \xrightarrow{s^*} y \Rightarrow \exists d_1 \in D \ \ s.t \ \ x_d \in V \ \ \forall \ d \geq d_1. \]

Since \( D \) is a directed set and \( d_0, d_1 \in D \)

\( \Rightarrow \exists d_2 \in D \ \ s.t \ d_2 \geq d_0 \& d_2 \geq d_1. \)

\( \Rightarrow x_d \in U \ \forall \ d \geq d_2 \) and \( x_d \in V \ \forall \ d \geq d_2 \Rightarrow U \cap V \neq \emptyset. \)

This is a contradiction.

Conversely,

Suppose that every \( s^* \)-convergent net in \( X \) has a unique \( s^* \)-limit point. To prove that \( X \) is an \( s^*-T_2 \)-space. Suppose not

\( \Rightarrow \ \exists x, y \in X, x \neq y \ \ s.t \ \ \forall U \in N_{s^*}(x) \ \ \text{and} \ \ \forall V \in N_{s^*}(y), U \cap V \neq \emptyset. \)

\( \Rightarrow (N_{s^*}(x), \subseteq) \) and \( (N_{s^*}(y), \subseteq) \) are directed sets by inclusion.

Let \( \rho = N_{s^*}(x) \times N_{s^*}(y) \). Define a relation \( \geq \) on \( \rho \) as follows:

\( \forall (U, V), (W, S) \in \rho, \ \text{we have} \ (U, V) \geq (W, S) \iff U \geq W \& V \geq S. \)

It is easy to verify that \((\rho, \geq)\) is a directed set.

Let \( (U, V) \in \rho \Rightarrow x \in U, y \in V \ & U \cap V \neq \emptyset. \)
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\[ \begin{align*} 
\therefore U \cap V \neq \emptyset & \Rightarrow \exists x_{(U,V)} \in U \cap V. \\
\text{Define } x : \rho \to X & \text{ by } : x(U,V) = x_{(U,V)} \forall (U,V) \in \rho. \\
\Rightarrow (x(U,V))_{(U,V) \in \rho} & \text{ is a net in } X. \text{ We will show that } (x(U,V))_{(U,V) \in \rho} \text{ is } \\
s^*\text{-convergent to both } x \text{ and } y. \\
\text{For if } U \in N_{s^*}(x) \text{ and } V \in N_{s^*}(y), \text{ then for each } (N,M) \in \rho \text{ s.t } \\
(N,M) \supseteq (U,V), \text{ we have } x(N,M) = x_{(N,M)} \in N \cap M \subseteq U \cap V. \\
\Rightarrow x_{(N,M)} \in U \text{ and } x_{(N,M)} \in V. \\
\Rightarrow x(U,V) \xrightarrow{s^*} x \text{ and } x(U,V) \xrightarrow{s^*} y. \\
\text{This is a contradiction. Thus } (X, \tau) \text{ is an } s^*-T_2\text{-space.} 
\end{align*} \]

2.12. Definition:
Let $X$ be a topological space and $A \subseteq X$. A point $x \in X$ is said to be $s^*$-limit point of $A$ iff every $s^*$-open set $U$ in $X$ containing $x$ contains a point of $A$ different from $x$.

2.13. Theorem:
Let $X$ be a topological space and $A \subseteq X$. Then:

1. A point $x \in X$ is an $s^*$-limit point of $A$ iff there is a net $(x_d)_{d \in D}$ in $A - \{x\}$ $s^*$-converging to $x$.
2. A set $A$ is $s^*$-closed in $X$ iff no net in $A$ $s^*$-converges to a point in $X - A$.
3. A set $A$ is $s^*$-open in $X$ iff no net in $X - A$ $s^*$-converges to a point in $A$.

Proof:
1) \[ \Rightarrow \]
Let $x$ be an $s^*$-limit point of $A$. To prove that $\exists$ a net $(x_d)_{d \in D}$ in $A - \{x\}$ such that $x_d \xrightarrow{s^*} x$.

Since $x$ is an $s^*$-limit point of $A$ \[ \Rightarrow \forall N \in N_{s^*}(x) , N \cap A - \{x\} \neq \emptyset. \]

$\therefore (N_{s^*}(x), \subseteq)$ is a directed set by inclusion.
Since \( N \cap A - \{x\} \neq \emptyset \), \( \forall N \in N_{s^*}(x) \Rightarrow \exists x_N \in N \cap A - \{x\} \).

Define \( x : N_{s^*}(x) \to A - \{x\} \) by \( x(N) = x_N \ \forall N \in N_{s^*}(x) \).

\[ (x_N)_{N \in N_{s^*}(x)} \] is a net in \( A - \{x\} \). To prove that \( x_N \xrightarrow{s^*} x \).

Let \( N \in N_{s^*}(x) \) to find \( d_0 \in D \) such that \( x_d \in N \ \forall d \geq d_0 \).

Let \( d_0 = N \Rightarrow \forall d \geq d_0 \Rightarrow d = M \in N_{s^*}(x) \).

\( x_d = x(d) = x(M) = x_M \in M \cap A - \{x\} \subseteq M \subseteq N \Rightarrow x_M \in N \).

\( \Rightarrow x_d \in N \ \forall d \geq d_0 \). Thus \( x_N \xrightarrow{s^*} x \).

Conversely,

Suppose that \( \exists \) a net \( (x_d)_{d \in D} \) in \( A - \{x\} \) such that \( x_d \xrightarrow{s^*} x \).

To prove that \( x \) is an \( s^* \)-limit point of \( A \). Let \( U \in N_{s^*}(x) \), since \( x_d \xrightarrow{s^*} x \Rightarrow \exists d_0 \in D \) such that \( x_d \in U \ \forall d \geq d_0 \).

But \( x_d \in A - \{x\} \ \forall d \in D \Rightarrow U \cap A - \{x\} \neq \emptyset \ \forall U \in N_{s^*}(x) \).

Thus \( x \) is an \( s^* \)-limit point of \( A \).

2) \( \Rightarrow \)

Let \( A \) be an \( s^* \)-closed in \( X \). To prove that \( \exists \) no net in \( A \) \( s^* \)-converges to a point in \( X - A \).

Suppose not \( \Rightarrow \exists \) a net \( (x_d)_{d \in D} \) in \( A \) s.t \( x_d \xrightarrow{s^*} x \) and \( x \in X - A \).

By (2.9) \( x \in s^* - cl(A) \). Since \( A \) is \( s^* \)-closed in \( X \), then \( s^* - cl(A) = A \Rightarrow x \in A \). But \( x \in X - A \Rightarrow (X - A) \cap A \neq \emptyset \),

this is a contradiction.

Thus no net in \( A \) \( s^* \)-converges to a point in \( X - A \).

Conversely,

Suppose that \( \exists \) no net in \( A \) \( s^* \)-converges to a point in \( X - A \).

To prove that \( A \) is \( s^* \)-closed. Let \( x \in s^* - cl(A) \), then by (2.9) \( \exists \) a net \( (x_d)_{d \in D} \) in \( A \) such that \( x_d \xrightarrow{s^*} x \). By hypothesis, we get every net in \( A \) \( s^* \)-converges to a point in \( A \).
\[ A \in A \Rightarrow s^* - cl(A) \subseteq A. \] Since \( A \subseteq s^* - cl(A) \Rightarrow A = s^* - cl(A) \)

\[ \Rightarrow A \text{ is } s^*-\text{closed}. \]

3) By (2) \( A \) is \( s^*\)-open in \( X \) iff \( X - A \) is \( s^*\)-closed in \( X \) iff no net in \( X - A \) \( s^*\)-converges to a point in \( A \).

2.14. Remarks: Let \( (x_d)_{d \in D} \) be a net in a topological space \( X \) and \( x \in X \). Then:

1) If \( x_d \xrightarrow{s^*} x \), then every subnet of \( (x_d)_{d \in D} \) \( s^*\)-converges to \( x \).

2) If every subnet of \( (x_d)_{d \in D} \) has a subnet \( s^*\)-convergent to \( x \),
   then \( x_d \xrightarrow{s^*} x \).

3) If \( x_d = x, \forall d \in D \), then \( x_d \xrightarrow{s^*} x \).

2.15. Theorem:

Let \( X \) and \( Y \) be topological spaces. A function \( f : X \to Y \) is an \( s^*\)-irresolute iff whenever \( (x_d)_{d \in D} \) is a net in \( X \) such that \( x_d \xrightarrow{s^*} x \), then \( f(x_d) \xrightarrow{s^*} f(x) \).

Proof: \( \Rightarrow \)

Suppose that \( f : X \to Y \) is an \( s^*\)-irresolute and \( (x_d)_{d \in D} \) be a net in \( X \) s.t \( x_d \xrightarrow{s^*} x \). To prove that \( f(x_d) \xrightarrow{s^*} f(x) \).

Let \( V \in N_{s^*}(f(x)) \), since \( f \) is \( s^*\)-irresolute, then by (2.3)

\[ \exists U \in N_{s^*}(x) \text{ s.t } f(U) \subseteq V. \] Since \( U \in N_{s^*}(x) \) and \( x_d \xrightarrow{s^*} x \).

\[ \Rightarrow \exists d_0 \in D \text{ s.t } x_d \in U, \forall d \geq d_0. \]

\[ \Rightarrow \exists d_0 \in D \text{ s.t } f(x_d) \in f(U) \subseteq V, \forall d \geq d_0. \]

\[ \therefore \forall V \in N_{s^*}(f(x)), \exists d_0 \in D \text{ s.t } f(x_d) \in V, \forall d \geq d_0. \]

Thus \( f(x_d) \xrightarrow{s^*} f(x) \).
Conversely,

To prove that \( f : X \to Y \) is \( s^\ast \)-irresolute. Suppose not, then by (2.3) \( \exists V \in N_{s^\ast}(f(x)) \) s.t \( U \in N_{s^\ast}(x), f(U) \not\subseteq V \).

\[ \forall U \in N_{s^\ast}(x), \exists x_U \in U \text{ s.t } f(x_U) \not\in V. \]

\[ (N_{s^\ast}(x), \subseteq) \text{ is a directed set by inclusion.} \]

Define \( x : N_{s^\ast}(x) \to X \) by: \( x(U) = x_U \) \( \forall U \in N_{s^\ast}(x) \).

\[ (x_U)_{U \in N_{s^\ast}(x)} \text{ is a net in } X. \]

To prove that \( x_U \xrightarrow{s^\ast} x \).

Let \( U \in N_{s^\ast}(x) \) to find \( d_0 \in D \) such that \( x_d \in U \) \( \forall d \geq d_0 \).

Let \( d_0 = U \Rightarrow \forall d \geq d_0 \Rightarrow d = N \in N_{s^\ast}(x) \).

i.e. \( N \geq U \Leftrightarrow N \subseteq U \).

\[ x(N) = x_N \in N \subseteq U \Rightarrow x_N \in U \forall d \geq d_0 \Rightarrow x_U \xrightarrow{s^\ast} x. \]

But \( (f(x_U)) \) does not \( s^\ast \)-converges to \( f(x) \), since \( f(x_U) \not\in V \forall U \in N_{s^\ast}(x) \). This is a contradiction. Thus \( f : X \to Y \) is an \( s^\ast \)-irresolute.

2.16. Theorem:

Let \( (x_d)_{d \in D} \) be a net in a topological space \( X \) and for each \( d \) in \( D \) let \( A_d \) be the set of all points \( x_{d_0} \) for \( d_0 \geq d \). Then \( x \) is an \( s^\ast \)-cluster point of \( (x_d)_{d \in D} \) if and only if \( x \) belongs to the \( s^\ast \)-closure of \( A_d \) for each \( d \) in \( D \).

Proof: \( \Rightarrow \)

If \( x \) is an \( s^\ast \)-cluster point of \( (x_d)_{d \in D} \), then for each \( d, A_d \) intersects each \( s^\ast \)-neighborhood of \( x \) because \( (x_d)_{d \in D} \) is frequently in each \( s^\ast \)-neighborhood of \( x \). Therefore \( x \) is in the \( s^\ast \)-closure of each \( A_d \).

Conversely,

If \( x \) is not an \( s^\ast \)-cluster point of \( (x_d)_{d \in D} \), then there is an \( s^\ast \)-neighborhood \( U \) of \( x \) such that \( (x_d)_{d \in D} \) is not frequently in \( U \). Hence for some \( d \) in \( D \), if \( d_0 \geq d \), then \( x_{d_0} \not\in U \), so that \( U \) and \( A_d \)
are disjoint. Consequently x is not in the s*-closure of A_d.

3. S*-Convergence Of Filters

3.1. Definition:
A filter ξ on a topological space X is said to s*-converge to x ∈ X (written ξ \(\xrightarrow{s^*} x\)) iff \(N_{s^*}(x) \subseteq \xi\).

3.2. Definition:
A filter ξ on a topological space X has x ∈ X as an s*-cluster point (written ξ ∝ x) iff each F ∈ ξ meets each N ∈ \(N_{s^*}(x)\).

3.3. Remark:
A filter ξ on a topological space X has x ∈ X as an s*-cluster point iff \(\bigcap\{s^*-cl(F) : F \in \xi\}\).

Proof: To prove that \(\xi \propto x \iff x \in \bigcap\{s^*-cl(F) : F \in \xi\}\).
\[
\begin{align*}
\therefore \xi \propto x & \iff \forall N \in N_{s^*}(x) \land \forall F \in \xi, N \cap F \neq \emptyset \\
& \iff \forall N \in N_{s^*}(x), F \cap N \neq \emptyset, \forall F \in \xi \\
& \iff x \in s^*-cl(F), \forall F \in \xi \\
& \iff x \in \bigcap\{s^*-cl(F) : F \in \xi\}.
\end{align*}
\]

3.4. Theorem:
Let X be a topological space and ξ be a filter on X and x ∈ X. Then :

1) If \(\xi \xrightarrow{s^*} x\), then \(\xi \propto x\).

2) If \(\xi \xrightarrow{s^*} x\), then \(\xi \rightarrow x\).

3) If \(\xi \propto x\), then \(\xi \propto x\).

4) If \(\xi \xrightarrow{s^*} x\), then every filter finer than ξ also s*-converges to x.
Proof: It is an obvious.

3.5. Remark:
The converse of (3.4) may not be true in general. To show that we
give the following examples:

Examples:

1) Let $(\mathfrak{R}, \mu)$ be the usual topological space where $\mathfrak{R}$ be the set
of all real numbers and $\xi = \{A \subseteq \mathfrak{R} : [-1,1] \subseteq A\}$ be a filter
on $\mathfrak{R}$, then $\xi \approx 0$, but $\xi$ does not $s^*$-converge to 0, since
$(-1,1) \in N_{s^*}(0)$, but $(-1,1) \notin \xi$.

2) Let $X = \{1,2,3\} \& \tau = \{\phi, X, \{1,2\}\}$
$\Rightarrow S^*O(X) = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$.
Let $\xi = \{X, \{1,2\}\}$ be a filter on $X$.
$\therefore N(1) = \{X, \{1,2\}\} \Rightarrow N(1) \subseteq \xi \Rightarrow \xi \rightarrow 1$.
$\therefore N_{s^*}(1) = \{X, \{1\}, \{1,2\}, \{1,3\}\} \Rightarrow N_{s^*}(1) \subsetneq \xi$
$\Rightarrow \xi$ is not $s^*$-converge to 1.

3) Let $X = \{1,2,3\} \& \tau = \{\phi, X\}$
$\Rightarrow S^*O(X) = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$.
Let $\xi = \{X, \{1,2\}\}$ be a filter on $X$.
$\therefore N(3) = \{X\} \Rightarrow \xi \approx 3$.
$\therefore N_{s^*}(3) = \{X, \{3\}, \{2,3\}, \{1,3\}\} \Rightarrow \xi$ is not $s^*$-cluster to 3, since
$\{3\} \cap \{1,2\} = \phi$.

4) Let $X = \{1,2\} \& \tau = \{\phi, X, \{1\}\}$ $\Rightarrow S^*O(X) = \{\phi, X, \{1\}\}$.
Let $\xi' = \{X, \{1\}\} \& \xi = \{X\}$.
$\therefore N_{s^*}(1) = \{X, \{1\}\} \Rightarrow N_{s^*}(1) \subseteq \xi' \Rightarrow \xi' \xrightarrow{s^*} 1$. 
3.6. Definition:
A filter base $\xi_0$ on a topological space $X$ is said to $s^*$-converge to $x \in X$ (written $\xi_0 \rightarrow_{s^*} x$) iff the filter generated by $\xi_0$ $s^*$-converges to $x$.

3.7. Definition:
A filter base $\xi_0$ on a topological space $X$ has $x \in X$ as an $s^*$-cluster point (written $\xi_0 \sim x$) iff each $F_0 \in \xi_0$ meets each $N \in N_{s^*}(x)$ (iff the filter generated by $\xi_0$ $s^*$-clusters at $x$).

3.8. Theorem:
A filter base $\xi_0$ on a topological space $X$ $s^*$-converges to $x \in X$ iff for each $N \in N_{s^*}(x)$, there is $F_0 \in \xi_0$ such that $F_0 \subseteq N$.

Proof: $\Rightarrow$
Given $\xi_0 \rightarrow_{s^*} x$, then the filter $\xi$ generated by $\xi_0$ $s^*$-converges to $x$. i.e. $\xi \rightarrow_{s^*} x \Rightarrow N_{s^*}(x) \subseteq \xi \Rightarrow \forall N \in N_{s^*}(x), N \in \xi$
$\Rightarrow \exists F_0 \in \xi_0$ s.t $F_0 \subseteq N$.

Conversely,
To prove that $\xi_0 \rightarrow_{s^*} x$ i.e. $\xi$ generated by $\xi_0$ $s^*$-converges to $x$. Let $N \in N_{s^*}(x)$, then by hypothesis, $\exists F_0 \in \xi_0$ s.t $F_0 \subseteq N$, since $\xi$ is a filter, then $N \in \xi \Rightarrow N_{s^*}(x) \subseteq \xi \Rightarrow \xi \rightarrow_{s^*} x$
$\Rightarrow \xi_0 \rightarrow_{s^*} x$.

3.9. Theorem:
A filter $\xi$ on a topological space $X$ has $x \in X$ as an $s^*$-cluster point iff there is a filter $\xi'$ finer than $\xi$ which $s^*$-converges to $x$. 

But $\xi \subseteq \xi'$ and $\xi$ is not $s^*$-converge to 1, since $N_{s^*}(1) \not\subseteq \xi$. 
Proof: \( \Rightarrow \)

If \( \xi \propto x \), then by (3.2) each \( F \in \xi \) meets each \( N \in N_{s^*}(x) \).

\[ \xi' = \{ N \cap F : N \in N_{s^*}(x), F \in \xi \} \]

is a filter base for some filter \( \xi' \) which is finer than \( \xi \) and \( s^* \)-converges to \( x \).

Conversely,

Given \( \xi \subseteq \xi' \) and \( \xi' \xrightarrow{s^*} x \Rightarrow \xi \subseteq \xi' \) and \( N_{s^*}(x) \subseteq \xi' \).

\[ \Rightarrow \text{each } F \in \xi \text{ and each } N \in N_{s^*}(x) \text{ belong to } \xi'. \]

Since \( \xi' \) is a filter \( \Rightarrow N \cap F \neq \phi \Rightarrow \xi \propto x \).

3.10. Theorem:

Let \( X \) be a topological space and \( A \subseteq X \). Then \( x \in s^* - \text{cl}(A) \) iff there is a filter \( \xi \) such that \( A \in \xi \) and \( \xi \xrightarrow{s^*} x \).

Proof: \( \Rightarrow \)

If \( x \in s^* - \text{cl}(A) \Rightarrow U \cap A \neq \phi \ \forall U \in N_{s^*}(x) \).

\[ \Rightarrow \xi_0 = \{ U \cap A : U \in N_{s^*}(x) \} \] is a filter base for some filter \( \xi \).

The resulting filter contains \( A \) and \( \xi \xrightarrow{s^*} x \).

Conversely,

If \( A \in \xi \) and \( \xi \xrightarrow{s^*} x \Rightarrow A \in \xi \) and \( N_{s^*}(x) \subseteq \xi \).

Since \( \xi \) is a filter \( \Rightarrow U \cap A \neq \phi \ \forall U \in N_{s^*}(x) \Rightarrow x \in s^* - \text{cl}(A) \).

3.11. Definition[8]:

Let \( X \) and \( Y \) be topological spaces, \( f : X \to Y \) be a function and \( \xi \) be a filter on \( X \), then \( f(\xi) \) is the filter on \( Y \) having for a base the sets \( \{ f(F) : F \in \xi \} \).

3.12. Theorem:

Let \( X \) and \( Y \) be two topological spaces. A function
\( f : X \rightarrow Y \) is an s*-irresolute iff whenever \( \xi \xrightarrow{s^*} x \) in \( X \), then \( f(\xi) \xrightarrow{s^*} f(x) \) in \( Y \).

**Proof:** \( \implies \)

Suppose that \( f : X \rightarrow Y \) is s*-irresolute and \( \xi \xrightarrow{s^*} x \).

To prove that \( f(\xi) \xrightarrow{s^*} f(x) \) in \( Y \). Let \( V \in N_{s^*}(f(x)) \), since \( f \) is s*-irresolute, then by (2.3), there is an s*-neighborhood \( U \) of \( x \) such that \( f(U) \subseteq V \). Since \( \xi \xrightarrow{s^*} x \), then \( U \in \xi \Rightarrow f(U) \in f(\xi) \).

But \( f(U) \subseteq V \), then \( V \in f(\xi) \). Thus \( f(\xi) \xrightarrow{s^*} f(x) \).

**Conversely,**

Suppose that whenever \( \xi \xrightarrow{s^*} x \) in \( X \), then \( f(\xi) \xrightarrow{s^*} f(x) \) in \( Y \).

To prove that \( f : X \rightarrow Y \) is s*-irresolute.

Let \( \xi = \{U : U \in N_{s^*}(x)\} \Rightarrow \xi \) is a filter on \( X \) and \( \xi \xrightarrow{s^*} x \).

By hypothesis \( f(\xi) \xrightarrow{s^*} f(x) \Rightarrow \) each \( V \in N_{s^*}(f(x)) \) belongs to \( f(\xi) \)

\( \Rightarrow \exists U \in N_{s^*}(x) \) s.t \( f(U) \subseteq V \Rightarrow f : X \rightarrow Y \) is an s*-irresolute function.

**3.13. Theorem:**

Let \( X \) be a topological space and \( A \subseteq X \). Then a point \( x \in X \) is an s*-limit point of \( A \) iff \( A - \{x\} \) belongs to some filter which s*-converges to \( x \).

**Proof:** \( \implies \)

If \( x \) is an s*-limit point of \( A \) \( \Rightarrow U \cap A - \{x\} \neq \emptyset \ \forall U \in N_{s^*}(x) \).

\( \Rightarrow \xi_0 = \{U \cap A - \{x\} : U \in N_{s^*}(x)\} \) is a filter base for some filter \( \xi \).

The resulting filter contains \( A - \{x\} \) and \( \xi \xrightarrow{s^*} x \).
Conversely,
If $A - \{x\} \in \xi$ and $\xi \xrightarrow{s^*} x \Rightarrow A - \{x\} \in \xi$ and $N_{s^*}(x) \subseteq \xi$.
Since $\xi$ is a filter $\Rightarrow U \cap A - \{x\} \neq \emptyset \ \forall U \in N_{s^*}(x)$. Thus $x$ is an $s^*$-limit point of a set $A$.

3.14. Definition[8]:
If $(x_d)_{d \in D}$ is a net in a topological space $X$, the filter generated by the filter base $\xi_0$ consisting of the sets $B_{d_0} = \{x_d : d \geq d_0\}, d_0 \in D$ is called the filter generated by $(x_d)_{d \in D}$.

3.15. Theorem:
A net $(x_d)_{d \in D}$ in a topological space $X$ $s^*$-converges to $x \in X$ iff the filter generated by $(x_d)_{d \in D}$ $s^*$-converges to $x$.

Proof: The net $(x_d)_{d \in D}$ $s^*$-converges to $x$ iff each $s^*$-neighborhood of $x$ contains a tail of $(x_d)_{d \in D}$, since the tails of $(x_d)_{d \in D}$ are a base for the filter generated by $(x_d)_{d \in D}$, the result follows.

3.16. Definition[8]:
If $\xi$ is a filter on a topological space $X$, let $D_\xi = \{(x, F) : x \in F \in \xi\}$. Then $D_\xi$ is directed by the relation $(x_1, F_1) \leq (x_2, F_2)$ iff $F_2 \subseteq F_1$, so the function $p : D_\xi \rightarrow X$ defined by $p(x, F) = x$ is a net in $X$. It is called the net based on $\xi$.

3.17. Theorem:
A filter $\xi$ on a topological space $X$ $s^*$-converges to $x \in X$ iff the net based on $\xi$ $s^*$-converges to $x$. 
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**Proof:**
Suppose that $\xi \xrightarrow{s^*} x$. If $N \in N_{s^*}(x)$, then $N \in \xi$. Since $N \neq \emptyset$, then $\exists p \in N$. Let $d_0 = (p, N) \in D_{\xi}$.

Thus the net based on $\xi$ $s^*$-converges to $x$.

Conversely, suppose that the net based on $\xi$ $s^*$-converges to $x$.

Let $N \in N_{s^*}(x)$, then $\exists d_0 = (p_0, F_0) \in D_{\xi}$ such that

$\forall d = (q, F) \geq d_0 = (p, N) \Rightarrow x_d = x_{(q, F)} = q \in F \subseteq N.$

3.18 Theorem:
A topological space $X$ is an $s^*$-$T_2$-space iff every $s^*$-convergent filter in $X$ has a unique $s^*$-limit point.

**Proof:**
Let $X$ be an $s^*$-$T_2$-space and $\xi$ be a filter in $X$ such that

$\xi \xrightarrow{s^*} x$ & $\xi \xrightarrow{s^*} y$ & $x \neq y$. Since $X$ is an $s^*$-$T_2$-space

$\Rightarrow \exists U \in N_{s^*}(x)$ and $V \in N_{s^*}(y)$ such that $U \cap V = \emptyset$.

$\therefore \xi \xrightarrow{s^*} x \Rightarrow N_{s^*}(x) \subseteq \xi$.

$\therefore \xi \xrightarrow{s^*} y \Rightarrow N_{s^*}(y) \subseteq \xi$.

$\therefore U \in N_{s^*}(x) \subseteq \xi$ & $V \in N_{s^*}(y) \subseteq \xi \Rightarrow U, V \in \xi$.

Since $\xi$ is a filter, then $U \cap V \neq \emptyset$. This is a contradiction. Hence $\xi$ $s^*$-converges to a unique $s^*$-limit point.

Conversely,
To prove that $X$ is an $s^*$-$T_2$-space. Suppose not, then

$\exists x, y \in X, x \neq y$ s.t $\forall U \in N_{s^*}(x)$ & $\forall V \in N_{s^*}(y)$, $U \cap V \neq \emptyset$.
\[ \xi_0 = \{ U \cap V : U \in N_{s^*}(x), V \in N_{s^*}(y) \} \] is a filter base for some filter \( \xi \). The resulting filter \( s^* \)-converges to \( x \) and \( y \).

This is a contradiction. Thus \( X \) is an \( s^*-T_2 \)-space.

**References**


حوَّل تقارب الشبكات والمرشحات - ُ

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المستخلص

كرس هذا البحث لتقديم و دراسة العديد من الخواص التبولوجية لمقاربة
الشبكات من النمط - ُ (convergence of nets) ُ وتقارب المرشحات من
النمط - ُ (convergence of filters) ُ مستخدمين مفهوم المجموعات
المفتوحة من النمط - ُ (open sets) ُ. كذلك درسنا بعض خواص النقاط
العنقودية من النمط - ُ (cluster points) ُ للشبكات والمرشحات.