RIGHT \((σ,τ)\)-DERIVATIONS ON LEFT IDEALS

Asawer D. Hamdi
asawerdurai@yahoo.com
Department of Mathematics, College of Science, University Of Baghdad. Baghdad-Iraq

Abstract

Let \( R \) be a prime ring and \( I \) a nonzero left ideal of \( R \) which is a semi prime as a ring.
For a right \((σ,τ)\)-derivations \( δ: R \to R \), we prove the following results:
(1) If \( δ \) acts as a homomorphism on \( I \), then \( δ = 0 \) on \( R \).
(2) If \( δ \) acts as an anti-homomorphism on \( I \), then either \( δ = 0 \) on \( R \) or \( I \subseteq Z(R) \).

Keywords: derivation, right derivation, \((σ,τ)\)-derivation, right \((σ,τ)\)-derivation

1. Introduction:
Throughout the present paper \( R \) will be denote an associative ring with center \( Z(R) \). Recall that \( R \) is prime if \( aRb=\{0\} \) implies that either \( a=0 \) or \( b=0 \). Let \( x, y \in R \), the commutator \([x,y]\) will denoted \( xy-yx \). An additive mapping \( d: R \to R \) is called a derivation (resp., Jordan derivation) on \( R \) if \( d(xy)=d(x)y+xd(y) \) (resp., \( d(x^2)=d(x)x+xd(x) \)) holds, for all \( x, y \in R \). Let \( σ,τ \) are two mappings of \( R \). An additive mapping \( d: R \to R \) is called a \((σ,τ)\)-derivation (resp., Jordan \((σ,τ)\)-derivation) on \( R \) if \( d(xy)=d(x)σ(y)+τ(x)d(y) \) (resp., \( d(x^2)=d(x)σ(x)+τ(x)d(x) \)) holds, for all \( x, y \in R \). Clearly every \((I,I)\)-derivation (resp., Jordan \((I,I)\)-derivation), where \( I \) is the identity mapping on \( R \) is derivation (resp., Jordan derivation) on \( R \). An additive mapping \( δ: R \to R \) is called a left derivation (resp., Jordan left derivation) if \( δ(xy)=x δ(y)+y δ(x) \) (resp., \( δ(x^2)=2xδ(x) \)) holds, for all \( x, y \in R \). In view of the definition of a \((σ,τ)\)-derivation the notion of left \((σ,τ)\)-derivation can be extended as follows:
An additive mapping \( δ: R \to R \) is called a left \((σ,τ)\)-derivation (resp., Jordan left \((σ,τ)\)-derivation) on \( R \) if \( δ(xy)=δ(x)σ(y)+τ(x)δ(y) \) (resp., \( δ(x^2)=σ(x)δ(x)+τ(x)δ(x) \)) holds, for all \( x, y \in R \). Clearly every left \((I,I)\)-derivation (resp., Jordan left \((I,I)\)-derivation) is a left derivation (resp., Jordan left derivation) on \( R \). An additive mapping \( δ: R \to R \) is called a right derivation (resp., Jordan right derivation) on \( R \) if \( δ(xy)=δ(y)x+δ(x)y \) (resp., \( δ(x^2)=2δ(x)x \)) holds, for all \( x, y \in R \).
An additive mapping \( δ: R \to R \) is called a right \((σ,τ)\)-derivation (resp., Jordan right \((σ,τ)\)-derivation) on \( R \) if \( δ(xy)=δ(y)σ(x)+δ(x)τ(y) \)
(resp., $\delta(x^2) = \delta(x)\sigma(x) + \delta(x)\tau(x)$) holds, for all $x, y \in R$. Clearly, every right $(1, 1)$-derivation (resp., Jordan right $(1, 1)$-derivation) on $R$ is a right derivation (resp., Jordan right derivation) on $R$.

Bell and Kappe [1] proved that if $d$ is a derivation of a prime ring $R$ which acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal $I$ of $R$, then $d=0$ on $R$. Further, Yenigul and Arac [2] obtained the above result for $\alpha$-derivation in prime rings. Recently, Ashraf, et al. [3] extended the result for $(\sigma, \tau)$-derivation in prime and semiprime rings. Also in [4] Ô.Glibasi and N. Aydin proved that if $d$ is a $(\sigma, \tau)$-derivation which acts homomorphism or as an anti-homomorphism on a prime ring $R$, then $d=0$ on $R$. In [5] Majeed and Hamdi Asawer extended the above results for $(\sigma, \tau)$-derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero Jordan ideal and a subring $J$ of a 2-torsion-free prime ring $R$, then they generalized the above extension for generalized $(\sigma, \sigma)$-derivation. Also they proved that if $d: R \to R$ is a $(\sigma, \tau)$-derivation which acts as a homomorphism on a nonzero Jordan ideal and a subring $J$ of a 2-torsion-free prime ring $R$, then either $d=0$ on $R$ or $J \subseteq Z(R)$.

In [6] Zaidi, et al. proved that if $R$ is a 2-torsion-free prime ring, $J$ a nonzero Jordan ideal and a subring of $R$ and $d$ is a left $(\sigma, \sigma)$-derivation of $R$, which acts as a homomorphism or as an anti-homomorphism on $R$, then $d=0$ on $R$. Hamdi Asawer in [7] extended this result to a left $(\sigma, \tau)$-derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero Jordan ideal and a subring $J$ of $R$.

As for more details and fundamental results used in this paper without mention we refer to [1,3,4,8,9,10,12]. The aim in this paper is to extend the above results and the theorem of Ô.Glibasi and N. Aydin [4] which state that if $d$ is a nonzero $(\sigma, \tau)$-derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero left ideal $I$ of prime ring $R$ which is a semiprime as a ring, then $d=0$ on $R$ to a right $(\sigma, \tau)$-derivation on $R$ which acts as a homomorphism or as an anti-homomorphism on a nonzero left ideal $I$ of prime ring $R$ which is a semiprime as a ring, then either $\delta=0$ on $R$ or $I \subseteq Z(R)$.

2. Right $(\sigma, \tau)$-derivation as a homomorphism or as an anti-homomorphism:

Let $R$ be a ring and $d$ is a derivation of $R$. If $d(xy)=d(x)d(y)$ (resp., $d(xy)=d(y)d(x)$) holds, for all $x, y \in R$, then we say that $d$ acts as a homomorphism (resp., anti-homomorphism) on $R$.

To prove the main result the following lemma is needed.

**Lemma (2.1):**

Let $R$ be a prime ring, $I$ a nonzero left ideal of $R$ which is semiprime as a ring. If $Ia=0$ ($aI=0$), for $a \in R$, then $a=0$.

We are now well-equipped to prove the main theorem:

**Theorem (2.2):**

Let $R$ be a prime ring, $I$ a nonzero left ideal of $R$ which is a semiprime as a ring. Suppose $\sigma, \tau$ are automorphisms of $R$ and $\delta: R \to R$ is a right $(\sigma, \tau)$-derivation of $R$. Then the following are holds:

(i) If $\delta$ acts as a homomorphism on $I$, then $\delta=0$ on $R$.

(ii) If $\delta$ acts as an anti-homomorphism on $I$, then either $\delta=0$ on $R$ or $I \subseteq Z(R)$.

**Proof:**

(i) If $\delta$ acts as a homomorphism on $I$, then we have $\delta(uv)=\delta(v)\sigma(u)+\delta(u)\tau(v)$, for all $u, v \in I$.

(2.1) Replacing $u$ by $ut$, $t \in I$ in (2.1), we get

$\delta(v)\sigma(ut)+\delta(ut)\tau(v)=\delta(ut)\delta(v)$

Since $\delta$ is a homomorphism on $R$ and $\sigma, \tau$ are automorphisms of $R$, we have

$\delta(u)\delta(v)=\delta(u)[\delta(v)\sigma(t)+\delta(t)v]$, for all $u, v, t \in I$.

Or equivalently

$\delta(v)\sigma(ut)=\delta(ut)\delta(v)$, for all $u, v, t \in I$ (2.2)

This implies that $[\delta(v)\sigma(u)-\delta(u)\delta(v)]\sigma(t)=0$, for all $u, v, t \in I$.

Hence $\sigma^{-1}([\delta(v)\sigma(u)-\delta(u)\delta(v)])I=\{0\}$, for all $u, v \in I$ and then we have $\sigma^{-1}([\delta(v)\sigma(u)-\delta(u)\delta(v)])RI=\{0\}$, for all $u, v \in I$, since $R$ is a prime ring and $I$ is a nonzero left ideal of $R$, we have $\delta(u)\sigma(u)-\delta(u)\delta(v)=0$, for all $u, v \in I$, since $\delta$ is a homomorphism on $R$, we get

$0=\delta(v)\sigma(u)-\delta(u)v$

$=\delta(v)\sigma(u)-\delta(v)\sigma(u)-\delta(u)\tau(v)$
\[ \delta(u)v = -\delta(u)\tau(v), \quad \text{for all } u,v \in I. \]

This implies that \( \delta(u)v = 0 \), for all \( u,v \in I \). Replacing \( v \) by \( rv \), \( r \in R \), we get
\[ 0 = \delta(u)(rv) = \delta(u)\tau(r)\tau(v), \quad \text{for all } u,v \in I, r \in R. \]
Since \( R \) is a prime ring and \( I \) is a nonzero left ideal of \( R \), we have \( \delta(u) = 0 \), for all \( u \in I \). Now, replacing \( u \) by \( ru \), \( r \in R \), we find
\[ 0 = \delta(ru) \]
\[ = \delta(u)\sigma(r) + \delta(r)\tau(u) = \delta(r)\tau(u), \quad \text{for all } u \in I, r \in R. \]
Since \( R \) is a prime ring, \( I \) a nonzero left ideal of \( R \) and \( \tau \) is an automorphism of \( R \), we have \( \delta = 0 \) on \( R \).

(ii) If \( \delta \) acts as an anti-homomorphism on \( I \), then we have \( \delta(uv) = \delta(v)\sigma(u) + \delta(u)\tau(v) = \delta(v)\delta(u) \), for all \( u,v \in I \).

Replacing \( u \) by \( uv \) in (2.3), we get
\[ \delta(v)\sigma(uv) + \delta(uv)\tau(v) = \delta(v)\delta(uv) \quad \text{for all } u,v \in I. \]

Since \( \delta \) is a homomorphism on \( R \) and \( \sigma, \tau \) are automorphisms of \( R \), we have\( \delta(v)\sigma(u)\sigma(v) + \delta(v)\delta(u)\tau(v) = \delta(v)\delta(u)\tau(v), \quad \text{for all } u,v \in I. \)

This implies that \( \delta(v)\sigma(u)\sigma(v) = \delta(v)\delta(u)\tau(v), \quad \text{for all } u,v \in I. \)

Replacing \( u \) by \( ut \), \( t \in I \) in (2.4), we get
\[ \delta(v)\sigma(u)\sigma(t)(v) = \delta(v)\delta(u)\sigma(t), \quad \text{for all } u,v,t \in I. \]

In view of (2.4), the relation (2.5) yields that
\[ \delta(v)\sigma(u)\sigma(t)(v) = \delta(v)\delta(u)\sigma(t), \quad \text{for all } u,v,t \in I, \]
this implies that \( \delta(v)\sigma(u)\sigma(t) = \delta(v)\delta(u) \), for all \( uv,t \in I \) and hence \( \sigma^{-1}(\delta(v))I \{ v,t \} = \{ 0 \}, \quad \text{for all } v,t \in I. \)

Since \( R \) is a prime ring, we have either \( \delta(v) = 0 \) or \( I \{ v,t \} = \{ 0 \}, \quad \text{for all } v,t \in I. \)

If \( \delta(v) = 0 \), for all \( v \in I \), replacing \( v \) by \( rv \), where \( r \in R \), to get \( \delta(rv) = \delta(v)\tau(r) + \delta(r)\tau(v) \), this implies that \( \delta(r)\tau(v) = 0 \), for all \( v \in I, r \in R. \)

Since \( R \) is a prime ring, \( I \) a nonzero ideal of \( R \) and \( \tau \) is an automorphism of \( R \), we have \( \delta = 0 \) on \( R. \)

If \( I \{ v,t \} = \{ 0 \} \) thus by Lemma (2.1), we find that \( I \{ v,t \} = \{ 0 \}, \quad \text{for all } v,t \in I. \)

Now, replacing \( v \) by \( rv \), where \( v \in I \) and \( r \in R \), we get
\[ 0 = [rv,t] \]
\[ = r[v,t] + [r,t]v \]
\[ = [r,t]v, \quad \text{for all } v,t \in I \text{ and } r \in R \text{ and hence we have } [R,I] = \{ 0 \}. \]

Since \( R \) is a prime ring, \( I \) a nonzero left ideal of \( R \), we have \( [R,I] = \{ 0 \} \), therefore we have \( I \subseteq Z(R). \)

References


10. Ashraf M. 2005. On left \( (\theta,\theta) - \) derivations of prime rings, \textit{Archivum Mathematicum (Brno), Tomus}, \textbf{41}:157-166.

