



PURE – SUPPLEMENTED MODULES

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Abstract.

Let R be an associative ring with identity and M be unital non zero right R -module. M is called H - supplemented module if given any submodule A of M there exist a direct summand submodule D of M such that $M = A+X$ iff $M = D+X$ where X is a submodule of M . In this paper we will give a generalization for H - supplemented which is called pure- supplemented module. An R - module M is called pure- supplemented module if given any submodule A of M there exists a pure submodule P of M such that $M = A+X$ iff $M = P+X$. Equivalently, for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} \ll \frac{M}{P}$ and $\frac{A+P}{A} \ll \frac{M}{A}$.

Key words: Small submodule, Supplementod module, Pure module, lifting module..

(المقاسات النقية المكتملة)

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الخلاصة

لنكن R حلقة تجميعية ذات عنصر محايد وليكن M مقاسا احاديا غير صفري ايمن معرفا على R المقاس الجزئي N من M يقال بأنه مكمل من النوع H اذا أعطينا مقاس جزئي A فيوجد مقاس جزئي مجموع مباشر D من M بحيث $M = A+X$ اذا $M = D+X$ في هذا البحث سنقوم بدراسة المقاسات النقية المكتملة حيث قدمنا هذا التعريف كتعميم لمفهوم المقاسات المكتملة من النوع H . يقال للمقاس M بأنه مكمل نقي اذا أعطينا مقاس جزئي A فيوجد مقاس جزئي نقي P من M بحيث $M = A+X$ اذا $M = P+X$

Introduction :

Let R be an associative ring with identity and M be a non zero unital right R -module. A submodule N of M is called a small submodule of M , denoted by $N \ll M$, if $N + L \neq M$ for any proper submodule L of M [1]. let U be a submodule of M , a submodule V of M is called supplement of U if V is minimal element in the set of submodules $L \leq M$ with $U+L = M$ equivalently $U+V = M$ and $U \cap V \ll V$. An R -module M is called supplemented if every submodule of M has supplement in M [2].

M is called H -supplemented module if given any submodule A of M there exist a direct summand D of M such that $M = A+X$ iff $M = D+X$ [3]. M is called weakly-supplemented module if for each submodule A of M there exists a submodule D of M such that $M = A+D$ and $A \cap D \ll M$ [2].

An R -module M is called lifting if for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M$ [2].

In this work we generalize the H -supplemented module, by introduce the pure-supplemented module. M is called pure-supplemented module if given any submodule A of M there exists a pure submodule P of M such that $M = A+X$ iff $M = P+X$. Equivalently, for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} \ll \frac{M}{P}$ and $\frac{A+P}{A} \ll \frac{M}{A}$.

Since every direct summand submodule is pure then it is clear that each H -supplemented module is pure-supplemented.

In this work, we review the concept of pure-supplemented module and we discuss some of the basic properties of this types of modules.

In section two we introduce the completely pure-supplemented module as a generalization of completely H -supplemented. We call a module M is completely pure-supplemented module if every direct summand of M is pure-supplemented.

1. Pure-supplemented modules

In this section we introduce the pure-supplemented module as a generalization of H -supplemented.

Definition 1.1:[2] Let M be an R -module. M is called H -supplemented module if given

any submodule A of M there exists a direct summand submodule D of M such that $M = A+X$ iff $M = D+X$.

Definition 1.2:[3] Let M be an R -module. P is called pure submodule of M if $KM \cap P = KP$ for every ideal K in R .

Remarks 1.3:[3]

- 1) Any direct summand submodule is pure submodule in M .
- 2) If $H \leq M$ and $K \leq H$ such that H is pure in M and K is pure in H then K is pure in M .
- 3) If A is pure submodule of M and K is pure submodule of N then $A \oplus K$ is pure of $M \oplus N$.

As a generalization of H -supplemented module, we introduce the pure-supplemented module.

Definition 1.4: Let M be a module. M is called pure-supplemented module if given any submodule A of M there exists a pure submodule P of M such that $M = A+X$ iff $M = P+X$.

Since every direct summand submodule is pure then it is clear that each H -supplemented module is pure-supplemented.

Remark 1.5:

- 1-Every hollow module is pure-supplemented module.
- 2-Every lifting module is pure-supplemented module.
- 3-Every P -supplemented is weakly supplemented.

Proof :

- 1) Since every hollow module is H -supplemented module then is pure-supplemented module.
- 2) Let A be submodule of M there exists $K \leq A$, $M = N \oplus K$ where $N \leq M$, and $N \cap A \ll M$ then $M = A+N$ iff $M = K+N$ where K is pure since K is direct summand submodule.
- 3) Let N be submodule of M , such that $N+X=M$, to show that $N \cap X \ll M$. Let $(N \cap X)+L=M$, since M pure-supplemented there exists a pure submodule P of M such that $N \cap X+L=M$, iff $M = P+L$ iff

$N \cap X + P \leq L$ then $N \cap X \leq L$ hence $L=M$ and

In this section we introduce the pure – lifting module as a generalization of lifting module.

Definition 1.6: Let M be a module . M is called pure– lifting module if for every submodule A of M there exists a pure submodule P of M , $P \leq A$ such that $M = P+X$ with $A \cap X \ll X$.

It is clear that every lifting and simesimple module are pure– lifting module .

Theorem 1.7: The following are equivalent for an R - module M .

- 1) M is pure– lifting module
- 2) Every submodule N of M can be written as $N=A+S$ where A is pure in M and $S \ll M$.
- 3) For every submodule N of M there exists a pure submodule A of N such that $M= A +K$ and $\frac{N}{A} \ll \frac{M}{A}$.

Proof :1→2) Let N be submodule of M , then $M = P+X$, $P \leq N$ pure in M and $N \cap X \ll X$, hence $N \cap X \ll M$.

$N=N \cap M = N \cap (P+X) = N \cap P + N \cap X = P + (N \cap X)$.take $A= P$, $S= N \cap X$.

2→3) Let N be a submodule of M by (2) $N=A+S$ where A is pure in M and $S \ll M$

suppose $\frac{M}{A} = \frac{N}{A} + \frac{L}{A}$.Then $\frac{M}{A} = \frac{A+S}{A} + \frac{L}{A}$, Thus $A+S+L=M$, by(2) since $S \ll M$ then $A+L=M$.

3→1) Let N be submodule of M ,there exists a pure submodule A of N such that $M= A +K$ and $\frac{N}{A} \ll \frac{M}{A}$ to prove that $N \cap K \ll K$. Suppose that $N \cap K + B = K$ where $B \leq K$ then $M= A +K = A + N \cap K + B$ thus

$$\frac{M}{A} = \frac{A + (N \cap K) + B}{A} = \frac{N \cap K + A}{A} + \frac{A + B}{A} = \frac{N}{A} + \frac{A + B}{A}, \text{ since } \frac{N}{A} \ll \frac{M}{A} \text{ then } \frac{A + B}{A} = \frac{M}{A} \text{ thus } A+B=M$$

then and hence $B=K$ thus $N \cap K \ll K$.

Proposition 1.8: Every pure – lifting is pure-supplemented module .

$N \cap X \ll M$.

Proof : let M be pure – lifting and A be a submodule of M suppose that $M=A+Y$ then $M=K+L$ where $K \leq A$ and K pure in M and $A \cap L \ll M$, now $A= A \cap M = A \cap (K+L) = K + A \cap L$ then Let $M= A+X = K+A \cap L + X$ since $A \cap L \ll M$ then $M=K+X$ thus $M=A+X$ since $K \leq L$ then M is pure- supplemented module .

Proposition 1.9: Let M be an R - module M is pure– supplemented module iff for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} \ll \frac{M}{P}$ and

$$\frac{A+P}{A} \ll \frac{M}{A} .$$

Proof : (⇒) Let M be a pure- supplemented module .and $A \leq M$ then there exist a pure submodule P such that

$M = A+X$ iff $M= P+X$. suppose that $\frac{A+P}{P} + \frac{L}{P} = \frac{M}{P}$ then

$$\frac{A+P+L}{P} = \frac{M}{P} \text{ thus } \frac{A+L}{P} = \frac{M}{P}$$

Then $A+L=M$ M is pure - supplemented , $A+L=M=P+X$, $P \leq L$, then $A+X \leq L$, then $M \leq L$

thus $L=M$ $\frac{L}{P} = \frac{M}{P}$ therefore $\frac{A+P}{P} \ll \frac{M}{P}$

similarly $\frac{A+P}{A} \ll \frac{M}{A}$.

(⇐) Let A be submodule of M ,then there exists a pure submodule P of M

such that $\frac{A+P}{P} \ll \frac{M}{P}$ and

$\frac{A+P}{A} \ll \frac{M}{A}$.If $M=A+X$ then

$$\frac{A+X}{P} = \frac{M}{P}, \text{ then } \frac{M}{P} = \frac{A+P}{P} + \frac{X+P}{P}$$

but $\frac{A+P}{P} \ll \frac{M}{P}$.Then $\frac{M}{P} = \frac{X+P}{P}$ thus

$M=X+P$.In the same way one can show that if $M= X+P$ then $M=A+X$.

Proposition 1.10 Let M be pure– supplemented module and A be a submodule of M .If for every pure submodule P of M ,

$\frac{A+P}{A}$ is pure in $\frac{M}{A}$ then $\frac{M}{A}$ is pure-supplemented module .

Proof : Let $\frac{N}{A} \leq \frac{M}{A}$, and let $\frac{M}{A} = \frac{N}{A} + \frac{X}{A}$ where $A \leq X$ then $M=N+X$ iff $M=P+X$ where P is pure in M , (M pure-supplemented) .Then $\frac{M}{A} = \frac{P+X}{A} = \frac{P+A}{A} + \frac{X}{A}$ by assamption $\frac{A+P}{A}$ is pure in $\frac{M}{A}$ hence $\frac{M}{A}$ is pure-supplemented.

Recall that a submodule A of R -module M is called fully invariant if for every $f \in \text{End}_R(M)$, $f(X) \subseteq X$. A module M is called dou module if every submodule is fully invariant. And M is called distributive iff for every submodules K, L, N , of M we have $N+(K \cap L) = (N+K) \cap (N+L)$ or $N \cap (K+L) = (N \cap K) + (N \cap L)$ [2].

Corollary1.11: Let M be a distributive pure-supplemented module then $\frac{M}{A}$ is pure-supplemented module for every submodule A of M .

Proof : Let D be direct summand of M , then $M = D \oplus K$ for some K submodule of M .

$\frac{M}{A} = \frac{D+A}{A} + \frac{K+A}{A}$ and $A = A + (D \cap K) = (A+D) \cap (A+K)$ (M is distributive) then

$\frac{M}{A} = \frac{D+A}{A} \oplus \frac{K+A}{A}$ hence $\frac{D+A}{A}$ is

direct summand of $\frac{M}{A}$,

then is pure in $\frac{M}{A}$ thus by proposition (1.10)

we get $\frac{M}{A}$ is pure-supplemented.

Corollary 1.12: Let A be a submodule of M and $eA \subseteq A$ for all $e^2 = e \in \text{End}_R(M)$ then $\frac{M}{A}$ is pure-supplemented .In particular for every fully

invariant submodule Y of M , $\frac{M}{Y}$ is pure-supplemented.

Proof :

Let D is a direct summand of M consider the projection map $e: M \rightarrow D$ then $e^2 = e \in \text{End}_R(M)$, $eA \subseteq A$ and

hence $eA = A \cap D$. Since D is a direct summand of M then $M = D \oplus K$, $K \leq M$ hence $A = (A \cap D)$

$\oplus (A \cap K)$ now $\frac{D+A}{A} = \frac{D \oplus (A \cap K)}{A}$ and

$\frac{K+A}{A} = \frac{K \oplus (A \cap D)}{A}$ hence

$M = D \oplus K = D + A + K + A = (D \oplus (A \cap K)) + K + A$

Then $\frac{M}{A} = \frac{D \oplus (A \cap K)}{A} + \frac{K+A}{A}$

Since $(D \oplus (A \cap K)) \cap K + A = (A \oplus K) \cap (A \cap D) = A$. Then

$\frac{M}{A} = \frac{D \oplus (A \cap K)}{A} \oplus \frac{K+A}{A}$.

Hence $\frac{K+A}{A}$ is direct summand of $\frac{M}{A}$ then

is pure in $\frac{M}{A}$ and by (prop. 1.10) $\frac{M}{A}$ is pure-supplemented .

2-Completely pure - supplemented Modules

We call a module M is completely pure-supplemented module .If every direct summand of M is pure-supplemented.

Proposition 2.1: Every lifting is completely pure-supplemented module .

Proof :

Let N is a direct summand of M , and $L \leq N$ since M is lifting then $L = K \oplus A$ where K is a direct summand of M and $A \ll M$. [2], Let $X \leq M$ with $N = L + X = K + A + X$ since A direct summand of L and $A \ll M$ then by (1.3,2) $A \ll L$. Then $N = K + X$ hence $N = L + X$ iff $N = K + X$.

An R -module M has PSP if the sum of any two pure submodule of M is pure [4]. .

Proposition 2.2

Let M be pure-supplemented module and M has PSP then M is completely pure-supplemented module .

Proof : Let N is a direct summand of M We show that N is pure-supplemented . $M=N\oplus K$, $K \leq M$ assume P is pure in M then by assumption $N+P$ pure in M ,

$M= N+P\oplus K$ then

$\frac{M}{K} = \frac{K+P}{K} + \frac{K+N}{K}$ then $\frac{M}{K}$ is pure-supplemented (prop. 1.10) , but $\frac{M}{K} \cong N$ then

N pure- supplemented module .

Proposition 2.3:If an R -module M has PSP and $M=M_1\oplus M_2$ is duo module ,then M is pure- supplemented iff M_1 and M_2 are pure-supplemented modules.

Proof : \Rightarrow) Since M_1 and M_2 are fully invariant submoduleo , hence M_1 and M_2 are pure- supplemented modules (coro.1.12).

\Leftarrow) Assume M_1 and M_2 are pure-supplemented modules and let $L \leq M$,then there exist a pure submodule P_1 of M such that $M_1 = (L \cap M_1) + X$ iff $M_1 = P_1 + X$ for any submodule X of M_1 . And there exist a pure submodule P_2 of M such that

$M_2 = (L \cap M_2) + Y$ iff $M_2 = P_2 + Y$ for any submodule Y of M_2 .

Claim $M=(P_1 \oplus P_2) + Z$ iff $M=L+Z$ for any submodule Z of M . Assume $M=(P_1 + P_2) + Z$ then $M_1 = P_1 + (M_1 \cap (P_2 + Z)) = (L \cap M_1) + (M_1 \cap (P_2 + Z)) = M_1 \cap [(L \cap M_1) + (P_2 + Z)]$ then $M_1 \leq (L \cap M_1) + Z$, since let $m_1 \in M$,

$m_1 = x + a_2 + z$ where $a_2 \in P_2$, $x \in L \cap M_1$, $z \in Z$

since $Z = (Z \cap M_1) \oplus (Z \cap M_2)$ then $z = z_1 + z_2$ where $z_1 \in Z \cap M_1$ and $z_2 \in Z \cap M_2$ clearly $m = x + z_1$ then

$M_1 \leq (L \cap M_1) + Z$.Similarly

$M_2 \leq (L \cap M_2) + Z$ then $M = (L \cap M_1) + (L \cap M_2) + Z$ by modularity

$M_1 = (L \cap M_1) + [M_1 \cap (L \cap M_2) + Z]$ then $M_1 = P_1 + [M_1 \cap (L \cap M_2) + Z] = M_1 \cap [P_1 + (L \cap$

$M_2) + Z]$ (modular law) then $M_1 \leq [P_1 + (L \cap M_2) + Z]$ hence $M_1 \leq P_1 + Z$.In the same way $M_2 \leq P_2 + Z$ then $M = (P_1 + P_2) + Z$, $P_1 + P_2$ is pure (sum of two pure is pure) PSP.

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