On n-Weakly Regular Rings

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الملخص

كتعميم للحلقات المنتظمة بضعف، نُعَرِّف الحلقات المنتظمة بضعف من الـنمط n علـى إنهـا لكـل $a \in N(R)$ فأن $a \in aRaR$ وتسمى هذه الحلقة حلقة منتظمة بضعف من النمط n. في هذا البحث أعطينـا خواص متنوعة للحلقات المنتظمة بضعف من النمط n وكذلك درسنا العلاقة بين تلك الحلقات والحلقات المختزلة بإضافة بعض أنواع الحلقات ومنها الحلقات من النمط NCI و MC2 و SNF .

ABSTRACT

As a generalization of right weakly regular rings, we introduce the notion of right n-weakly regular rings, i.e. for all $a \in N(R)$, $a \in aRaR$. In this paper, first give various properties of right n-weakly regular rings. Also, we study the relation between such rings and reduced rings by adding some types of rings, such as NCI, MC2 and SNF rings.

Introduction:

Throughout this paper a ring *R* denotes an associative ring with identity and all modules are unitary. For a subset *X* of R, the left(right) annihilator of *X* in *R* is denoted by *r(X)(l(X))*. If *X={a}*, we usually abbreviate it to *r(a)(l(a))*. We write *J(R), and N(R),* for the Jacobson radical and the set of nilpotent elements respectively.

The center of the ring R is denoted by *Cent(R)* and it is $Cent(R) = \{a \in R / ar = ra \quad \forall \quad r \in R\}$. A ring R is called n-regular if for all $a \in N(R)$, $a \in aRa$ [7]. A right *R*-module *M* is called *N flat* if for any $a \in N(R)$, the mapping $1_M \otimes i : M \otimes_R Ra \to M \otimes_R R$ is monic, where $i : Ra \to R$ is the inclusion mapping [8]. A ring R is called right (left) SNF if every simple right(left) R-module is N flat [8].

A ring R is called *semiprime* if it has no nilpotent ideals [6]. A ring R is called *reduced* if *N(R)* = 0 [6], or equivalently, $a^2=0$ implies $a=0$ in R for all $a \in R$. Recall that a ring R is *MERT*(resp. *MELT*), if every maximal essential right (resp. left) ideal of R is an ideal [9].

2. n -Weakly Regular Ring

This section is devoted to give the definition of n-weakly regular rings with some of its characterizations and basic properties.

A ring R is *right (left) weakly regular* [6], if $a \in aRaR$ (RaRa) for every $a \in R$. We called R is weakly regular if it is both right and left weakly regular.

Definition 2.1

A ring R is to be *right* (left) *n-weakly regular* if $a \in aRaR$ ($a \in RaRa$) for all $a \in N(R)$. We say that R is n-weakly regular if it is right and left n-weakly regular ring.

Clearly every weakly regular rings are *n-weakly regular*.

Examples:

1- Every reduced ring is n-weakly regular.
2- Every n-regular ring is n-weakly regular ring.
3- The ring Z_6 of integers modulo 6, is reduced, n-regular, weakly regular ring, so it is n- weakly regular.
4- Let $R=\left\{\begin{bmatrix} Z_2 & Z_2 \\ Z_2 & Z_2 \end{bmatrix}\right\}$, $N(R)=\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$. *R is* n-regular, weakly regular ring, so it is n- weakly regular but R not reduced ring.
5- The ring *Z* of integer number is reduced, n-regular, so it is n- weakly regular but Z is not weakly regular ring.

Proposition 2.2

R is a right n-weakly regular ring if and only if aR is idempotent right ideal for all $a \in N(R)$.

Proof:

Let R is a right n-weakly regular ring and I is a principal right ideal of R generated by a nilpotent element, then there exists $a \in N(R)$ such that I=aR, clearly $I^2 \subseteq I$.

On the other hand, since R is right n-weakly, then there exists $y,z \in R$ such that a=ayaz. Now let $x \in I$, then there exists $r \in R$ such that $x=ar=ayazr \in I^2$. Therefore $I \subseteq I^2$. Hence $I^2 = I$.

Conversly, Let $a \in N(R)$, since aR is idempotent right ideal of R, so $a \in aR=aRaR$. Therefore R is right n-weakly regular ring. ■

Proposition 2.3

Let R be a right n-weakly regular ring. If $aR \subseteq I$, for $a \in N(R)$ and I is right or left ideal. Then aRI=aR.

Proof:

It is clearly that $bRI \subseteq bR$ for any $b \in R$. Now let $a \in N(R)$ and $x \in aR$, then there exists $r \in R$ such that x=ar. Since R is right n-weakly regular ring then there exists $y,z \in R$ such that a=ayaz, x=ayazr, hence $azr \in aR \subseteq I$. So $aR \subseteq aRI$. Therefore aRI=aR. ■

Corollary 2.4

Let R be a ring for all $a \in N(R)$ and any I right or left ideal of R such that $aR \subseteq I$. Then the following condition are equivalent:

1- R is right n-weakly regular.
2- For all $a \in N(R)$, aRI=aR.

Proof:

$1 \rightarrow 2$ by Proposition 2.3.
$2 \rightarrow 1$ Let I=aR and by Proposition 2.2. ■

Proposition 2.5

Let *R* be a right n-weakly regular ring. Then $N(R) \cap Cent(R)=0$

Proof :

If $N(R) \cap Cent(R) \neq 0$, then there exists $0 \neq a \in N(R) \cap Cent(R)$ such that $a^2=0$. since R is right n-weakly regular then there exists $y,z \in R$ such that $a=ayaz=a^2yz=0yz=0$ ($a \in Cent(R)$). Therefore $a=0$. This is shows that $N(R) \cap Cent(R)=0$. ■

Corollary 2.6

Let R be a commutative ring. Then R is reduced if and only if R is right n-weakly regular.

Lemma 2.7 [2]

1- Every one sided or two sided nil ideal of R is contained in J(R).
2- In any ring R, $a \in J(R)$ if and only if 1-ar is invertible for all $r \in R$.

Now we have the following proposition

Theorem 2.8

Let R be a right n-weakly regular ring. Then $N(R) \cap J(R)=0$.

Proof:

If $a \in N(R) \cap J(R)$, then there exists $y,z \in R$ such that a= ayaz. Hence a(1-yaz)=0, since $a \in J$, yaz $\in J$, then by Lemma 2.7(2), there exists invertible element v $\in R$ such that (1-yaz)v=1. So (a-ayaz)v=a, yield a=0. Therefore $N(R) \cap J(R)=0$. ■

Let R be a ring we denoted to the upper nil radical for a ring R by $Nil^*(R)$ and it is the sum of nil ideal in the ring R.

Corollary 2.9

Let R be a right n-weakly regular ring .Then $Nil^*(R)=0$.

Proof:

Let I be a left or right or two sided nil ideal, by Lemma 2.7(1), we have that $I \subseteq J(R)$, $I \subseteq N(R) \cap J(R)=0$, Theorem 2.8, I=0, which is a contradiction. So R not contain any nil ideal. Therefore $N^*(R)=0$. ■

Corollary 2.10

Let R be a right n-weakly regular ring. Then R is semiprime ring.

Proof:

Let I be a nilpotent right ideal then $I \subseteq N(R) \cap J(R)=0$ (Theorem 2.8*)* I=0. Therefore R is semiprime ring. ■

3. The Connection between n-Weakly Regular Rings and Other Rings.

In this section we gives the connection between n-weakly regular rings and reduced rings, SNF rings.

Proposition 3.1

The following conditions are equivalent for a ring R.

1- R is reduced.
2- R is right n-weakly regular and N(R) forms a right ideal of R.
3- R is right n-weakly regular and N(R) forms a left ideal of R.
4- R is right n-weakly regular and NI ring.
5- R is right n-weakly regular and $N(R) \subseteq J(R)$.

Proof:

$1 \to 4 \to 3 \to 5$, $1 \to 2 \to 5$ it is trivial.

$5 \to 1$

Suppose that R is right n-weakly regular ring. So $N(R) \cap J(R)=0$, (Theorem 2.8). Since $N(R) \subseteq J(R)$, then $N(R) \cap J(R)= N(R)=0$. Therefore R is reduced. ■

Theorem 3.2

Let R be a ring with aR=Ra, for all $a \in N(R)$. Then the following conditions are equivalent:

1- R is right n-weakly regular.
2- R is n-regular.
3- R is reduced.

Proof:

$1 \to 2$

Let $0 \neq a \in R$, such that $a^2=0$. Since R is a right n-weakly regular, then $a \in aR=aRaR$ (Proposition 2.1)
= aRRa (Ra=aR)
=aRa.
so $a \in aRa$, hence R is n-regular.

$2 \to 3$

Let $0 \neq a \in R$, such that $a^2=0$. Since R is a right n-regular, then there exists $b \in R$ such that a=aba since aR=Ra there exists $x \in R$ such that ab=xa, so $a=aba=xa^2=x0=0$. Therefore R is reduced.

$3 \to 1$ It is trivial. ■

A ring R is called NCI provided that N(R) contains a non zero ideal of R whenever $N(R) \neq 0$ [1].

Lemma 3.3 [1]

Let R be a ring with $N(R) \neq 0$. Then R is NCI if and only if $N^*(R) \neq 0$.

Proportion 3.4

Let R be a NCI ring , then R is right n-weakly regular if and only if R is reduced.

Proof:

Let $N(R) \neq 0$, sinc R is NCI ring from Lemma 3.3, we get that $N^*(R) \neq 0$ but R is right n-weakly regular, $N^*(R)=0$ (Corollary 2.9), which is contradiction. So N(R)=0. Therefore R is reduced. ■

A ring R is called weakly reversible if and only if for all $a,b,r \in R$ such that ab=0, Rbra is a nil left ideal of R (equivalently braR is nil right ideal of R). Clearly ZI ring is weakly reversible [4].

Proposition 3.5

A ring R be right n-weakly regular ring and weakly reversible if and only if R is reduced.

Proof:

Let $a \in R$ with $a^2=0$. Then a=ayaz for some $y,z \in R$ because R is right n-weakly regular ring. Since R is weakly reversible then ayaR is nil right ideal of R so ayaR$\subseteq J(R) \cap N(R) = 0$ (Lemma 2.7(1) & Theorem 2.8) we get ayaR=0, in particular a=ayaz=0. Therefore R is reduced.

Converse, it is trivial. ■

Recall that a ring R is right *MC2* if $K \cong eR$ is simple, $e^2=e$, , then K=gR for some $g^2=g$ [5].

Lemma 3.6 [9]

Let R be a left MC2,right SNF ring and MELT ring. Then R is a semiprime ring and right non singular.

Theorem 3.7 [9]

Let I be a right ideal of a ring R. Then R/I is N flat if and only if Ia=I$\cap$Ma for all a$\in$N(R).

Theorem 3.8

Let R be right SNF, left MC2 and MELT ring. Then R is left n-weakly regular ring.

Proof:

From Lemma 3.6, we get that R is a semiprime ring and a$\in$ N(R). If RaR+l(a) $\neq$R, then there exists a maximal left ideal M of R containing RaR+l(a), if M is not essential then we can write M=l(e), where e^2=e $\in$R and e$\neq$0, since RaRe=0 because RaR$\subseteq$M, $(ReRa)^2$=0 implies ReRa=0 (since R is semiprime) ReRa=0 in particular ea=0 and e$\in$l(a) $\subseteq$M=l(e), then e^2=0, which is a contradiction. Therefore M is an essential, since R is MELT ring, then M is a two sided ideal then there exists a maximal right ideal L in R containing M, since R is right SNF ring then R/ L is N flat right R-module, a=ma for some m$\in$M (Theorem 3.7), (1-m)a=0, 1-m$\in$l(a) $\subseteq$M$\subseteq$L therefore 1-m$\in$ L implies 1$\in$ L which is a contradiction, therefore RaR+l(a) =R for all a$\in$N(R). Thus R is left n-weakly regular ring. ■

Theorem 3.9

Let R be MELT and right SNF ring with RaR is essential for all a$\in$N(R), Then R is left n-weakly regular ring.

Proof:

Let a $\in$ N(R). If RaR+l(a) $\neq$R then there exists a maximal left ideal M of R containing RaR+l(a), since RaR is left annihilator of a nilpotent element by the hypothesis RaR is essential left ideal in R, M is an essential ideal of R (MELT ring), there exists a maximal right ideal L in R such that M$\subseteq$L. Since R is right SNF ring, then R/ L is N flat right R-module, a=ma for some m$\in$M (Theorem 3.7), 1-m$\in$l(a) $\subseteq$M$\subseteq$L implies 1$\in$ L, which is a contradiction. Therefore RaR+l(a) =R for all a$\in$N(R), and so R is left n-weakly regular ring. ■

Definition 3.10 [8]

A right *R*-module *M* is said to be *nil*-injective, if for any $a \in N(R)$, any right *R*-homomorphism $f: aR \rightarrow M$ can be extended to $R \rightarrow M$, or equivalently *f*=*m.,* where $m \in M$.

The ring R is called right nil-injective if R_R is right nil-injective. Clearly a reduced ring is a right nil-injective and n-regular ring is a right nil-injective [8].

Theorem 3.11

Let R be a semiprime ring whose simple singular right R-module are nil-injective. Then R is right n-weakly regular ring.

Proof:

Let a$\in$N(R). We claim that RaR+r(a)=R if not, there exists a maximal right ideal M of R containing RaR+r(a). If M is not essential in R then M=r(e), e^2=e$\in$R. Since Rae$\subseteq$RaR$\subseteq$M=r(e), eRae=0, $(aeR)^2$=0 but R is semiprime, aeR=0, so ae=0. Thus e $\in$r(a) $\subseteq$M=r(e), which is a contradiction. Hence M is essential right ideal in R and so R/M is nil-injective. Define a mapping $f: aR \rightarrow R/M$ such that f(ar)=r+M, let x,y$\in$R such that ax=ay, a(x-y)=0, (x-y)+M=f(a(x-y))=f(0)=M, then x-y$\in$M, x+M=y+M, f(ax)=x+M=y+M=f(ay), f is well define. 1+M=f(a)=(b+M)(a+M)=ba+M, 1-ba$\in$M, since ba$\in$RaR$\subseteq$M then 1$\in$M which is a contradiction, so RaR+r(a)=R, in particular there is y,z$\in$R and v$\in$r(a) such that yaz+v=1, ayaz+av=a, a=ayaz. Therefore R is right n-weakly regular ring. ■

Proposition 3.12

Let R be a ring whose simple right R-module are nil-injective. Then R is right n-weakly regular ring.

Proof:

Assume that a$\in$R such that aRa=0. Then RaR$\subseteq$r(a). If a$\neq$0 then there exists a maximal right ideal M containing r(a). By hypothesis R/M is nil-injective. We define a mapping $f: aR \to R/M$ such that f(ar)=r+M, f is well define similar to Theorem 3.11, so there exists b$\in$R such that 1+M=f(a)=ba+M, 1-ba$\in$M because ba$\in$RaR$\subseteq$M then 1$\in$M which is a contradiction, so a=0. Therefore R is a semiprime ring, by Theorem 3.11 we get that R is right n-weakly regular ring. ■

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