The n-Hosoya Polynomial of $W_\alpha \boxtimes C_\beta$

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الملخص

البيان المركب $W_\alpha \boxtimes C_\beta$ من بيان العجلة W_α وبيان الدارة C_β هو البيان الناتج من إتحاد W_α مع C_β وبإضافة الحافات u_1u_2 ، u_1v_2 ، v_1u_2 ، v_1v_2 ، إذ أن u_1v_1 حافة في W_α وأن u_2v_2 حافة في C_β . في هذا البحث تم إيجاد القطر – n ومتعددة حدود هوسويا – n ودليل وينر – n للبيان $W_\alpha \boxtimes C_\beta$.

ABSTRACT

For a wheel W_α and a cycle C_β the composite graphs $W_\alpha \boxtimes C_\beta$ is constructed from the union of W_α and C_β and adding the edges u_1u_2, u_1v_2, v_1u_2 and v_1v_2 , where u_1v_1 is an edge of W_α and u_2v_2 is an edge of C_β. The n – diameter , the n – Hosoya polynomial and the n – Wiener index of $W_\alpha \boxtimes C_\beta$ are obtained in this paper.

Keywords: n-distance, n-Hosoya polynomial, n-Wiener index, Wheel and cycle.

1. Introduction:

For the definitions of concepts and notations used in this paper, see the references [6, 7 and 8]. Some authors defined the minimum distance between two nonempty subsets of vertices of a connected graph G by [7]:

$d_{min}(A,B) = \min\{d(a,b): a \in A, b \in B\}$,

where A and B are nonempty subsets of vertices of a connected graph G.

<u>The n-distance</u> in a connected graph G = (V,E) [4] is the minimum distance from a singleton vertex $v \in V$ to an (n-1)-subset S , $S \subseteq V$, that is

$d_n(v,S) = \min\{d(v,u): u \in S\}$, $2 \le n \le p$, in which p is the order of G.

It is clear that :

$d_n(v,S) = 0$; when $v \in S$,

$d_n(v,S) \ge 1$; when $v \notin S$.

When n=2 , we get the (ordinary) distance d(u,v).

<u>The n-diameter of G</u> is defined by

$$\delta_n = \delta_n(G) = \max\{d_n(v,S): v \in V(G), |S| = n-1, S \subseteq V(G)\}. \qquad \ldots(1.1)$$

<u>The n-Hosoya polynomial of G of order p</u> is defined by

$$H_n(G;x) = \sum_{k=1}^{\delta_n} C_n(G,k)x^k, \qquad \ldots(1.2)$$

where $C_n(G,k)$ is the number of order pairs $(v,S), v \in V(G), S \subseteq V(G), |S| = n-1$, such that $d_n(v,S) = k$, $2 \le n \le p$.

One can easily show that [4].

$$C_n(G,1) = p\binom{p-1}{n-1} - \sum_{v\in V(G)}\binom{p-1-\deg v}{n-1}. \qquad \ldots(1.3)$$

<u>The n-Hosoya polynomial of a vertex v in G</u>, denoted by $H_n(v,G;x)$, is defined [4] by

$$H_n(v,G;x) = \sum_{k\geq 1} C_n(v,G,k)x^k, \qquad \ldots(1.4)$$

where $C_n(v,G,k)$ is the number of (n-1)-subsets of vertices S such that $d_n(v,S) = k$. It is clear that

$$C_n(G,k) = \sum_{v\in V(G)} C_n(v,G,k), \quad \text{for } 1 \leq k \leq \delta_n \qquad \ldots(1.5)$$

and

$$H_n(G;x) = \sum_{v\in V(G)} H_n(v,G;x), \qquad \ldots(1.6)$$

<u>The n-Wiener index of G</u> is defined by

$$W_n(G) = \frac{d}{dx}H_n(G;x)\Big|_{x=1} = \sum_{k=1}^{\delta_n} k\,C_n(G,k).$$

In [2], H. G. Ahmed gave the following result :

Lemma: Let v be any vertex of a connected graph G. If there are t_1 vertices of distance $k \geq 1$ from v , and there are t_2 vertices of distance more than k from v , then

$$C_n(v,G,k) = \binom{t_1+t_2}{n-1} - \binom{t_2}{n-1}. \# \qquad \ldots(1.7)$$

Definition(1): [1]

Let G_1 and G_2 be disjoint connected graphs , and let u_1v_1 be an edge of G_1 and u_2v_2 be an edge of G_2, then the composite graph $G_1 \boxtimes G_2$ is the graph constructed from G_1 and G_2 by adding the edges u_1u_2, u_1v_2, v_1u_2 and v_1v_2. It is clear that $p(G_1 \boxtimes G_2) = p(G_1)+p(G_2)$ and $q(G_1 \boxtimes G_2) = q(G_1)+q(G_2)+4$.

To simplify our discussion , we give the following :

Definition(2):

For every vertex v of a connected graph G and each k , $1 \leq k \leq \delta_n$, we define : $N_k^{=}(v)$ is the number of vertices w of G such that $d(v,w) = k$, and $N_k^{+}(v)$ is the number of vertices w of G such that $d(v,w) > k$.

Finally , it seems to us that it is impossible to obtain the n – Hosoya polynomial of $G_1 \boxtimes G_2$ for any disjoint connected graphs G_1 and G_2 in terms of $H_n(G_1;x)$ and $H_n(G_2;x)$. Therefore, in [5], A. M. Ali obtained $H_n(G_1 \boxtimes G_2;x)$ where G_1 is a complete graph and G_2 is a special graph such as a complete graph, a complete bipartite, a wheel, or a cycle and in [3]; H.G. Ahmed obtained $H_n(G_1 \boxtimes G_2;x)$ where G_1 and G_2 are wheels W_α and W_β. In the continuation of such work , we take G_1 as a wheel W_α and

G_2 as a cycle C_β and find the n – diameter, n – Hosoya polynomial and n – Wiener index of $W_\alpha \boxtimes C_\beta$.

2. The n-Hosoya Polynomial of $W_\alpha \boxtimes C_\beta$:

The graph $G = W_\alpha \boxtimes C_\beta$, for $\alpha \geq 6$ and $\beta \geq 4$ is shown in Fig. 2.1. It is clear that $p(G) = \alpha + \beta$, $q(G) = 2\alpha + \beta + 2$ and $\operatorname{diam} G = \left\lfloor \frac{\beta - 1}{2} \right\rfloor + 3$.

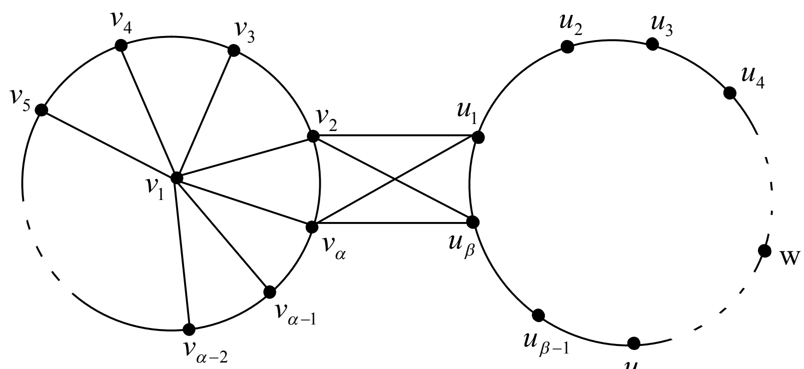

Fig. 2.1. The graph $G = W_\alpha \boxtimes C_\beta$, $\alpha \geq 6$ and $\beta \geq 4$

Let $V = V(W_\alpha) = \{v_1, v_2, \ldots, v_\alpha\}$ and $U = V(C_\beta) = \{u_1, u_2, \ldots, u_\beta\}$.

The following proposition determines the n-diameter of $G = W_\alpha \boxtimes C_\beta$.

Proposition 2.1: For $2 \leq n \leq p(= \alpha + \beta)$, $\alpha \geq 6$ and $\beta \geq 4$, we have

$$\operatorname{diam}_n(G) = \begin{cases} \left\lfloor \frac{\beta-1}{2} \right\rfloor + 3, & \text{for } 2 \leq n \leq \alpha - 4, \\ \left\lfloor \frac{\beta-1}{2} \right\rfloor + 2, & \text{for } \alpha - 3 \leq n \leq \alpha - 1, \\ \left\lfloor \frac{\beta-1}{2} \right\rfloor + 1, & \text{for } \alpha \leq n \leq \alpha + 1, \\ \left\lfloor \frac{\alpha+\beta-n}{2} \right\rfloor + 1, & \text{for } \alpha + 2 \leq n \leq \alpha + \beta. \end{cases}$$

Proof: For $2 \leq n \leq \alpha - 4$, we take $S \subseteq \{v_4, v_5, \ldots, v_{\alpha-2}\}$ and $w = u_{\beta/2}$ (or $w = u_{\beta/2+1}$) for even β and $w = u_{(\beta+1)/2}$ for odd β. Then, $d_n(w,S) = \left\lfloor \frac{\beta-1}{2} \right\rfloor + 3$ which is max. of such values of n .

For $\alpha - 3 \leq n \leq \alpha - 1$, we take $S \subseteq \{v_1, v_3, v_4, \ldots, v_{\alpha-1}\}$ with $|S| = n - 1$, then $d_n(w,S) = \left\lfloor \frac{\beta-1}{2} \right\rfloor + 2 = \operatorname{diam}_n G$.

Similarly , for $\alpha \le n \le \alpha+1$, we take $|S| = n-1$ and $S \subseteq V(W_\alpha)$, then $d_n(w,S) = \left\lfloor \frac{\beta-1}{2} \right\rfloor + 1 = \mathrm{diam}_n G$. Finally , for $\alpha+2 \le n \le \alpha+\beta$, we have $2 \le n-\alpha \le \beta$ $\mathrm{diam}_n(G) = \mathrm{diam}_{n-\alpha}(C_\beta) = \left\lfloor \frac{\beta-(n-\alpha)}{2} \right\rfloor + 1 = \left\lfloor \frac{\alpha+\beta-n}{2} \right\rfloor + 1$. In this case, we take S containing $V(W_\alpha)$ with the sequence of vertices $u_1, u_\beta, u_2, u_{\beta-1}, \ldots$ and so on to $\beta-1$ of vertices of $V(C_\beta)$. #

Since $H_n(G;x) = \sum_{k=1}^{\delta_n} C_n(G,k)x^k$, where δ_n is the n-diameter of G determined by Proposition 2.1, we shall find $C_n(G,k)$, $1 \le k \le \delta_n$ in order to get $H_n(G;x)$.
Using (1.3) , we get

$$C_n(G,1) = p\binom{p-1}{n-1} - (\alpha-3)\binom{p-4}{n-1} - (\beta-2)\binom{p-3}{n-1} - 2\binom{p-5}{n-1} - 2\binom{p-6}{n-1} - \binom{\beta}{n-1} \quad ...(2.1)$$

Proposition 2.2: For $2 \le n \le p$ and $\alpha \ge 6$, $\beta \ge 4$, we have

$$C_n(G,2) = (\beta-2)\binom{p-3}{n-1} + (\alpha-3)\binom{p-4}{n-1} + 2\binom{p-6}{n-1} - (\alpha-6)\binom{\beta}{n-1} - 3\binom{\beta-2}{n-1}$$
$$- 2\binom{\beta-4}{n-1} - (\beta-6)\binom{p-5}{n-1} - 2\binom{p-7}{n-1} - 2\binom{p-10}{n-1} \quad \ldots(2.2)$$

Proof: From Fig. 2.1, it is clear that $N_2^{=}(v_1) = 2$ and $N_2^{+}(v_1) = \beta-2$. Thus, by (1.7)

$$C_n(v_1, G, 2) = \binom{\beta}{n-1} - \binom{\beta-2}{n-1}. \quad \ldots(2.3)$$

$N_2^{=}(v_2) = \alpha-2$ and $N_2^{+}(v_2) = \beta-4$. Thus , by (1.7)

$$C_n(v_2, G, 2) = C_n(v_\alpha, G, 2) = \binom{p-6}{n-1} - \binom{\beta-4}{n-1}. \quad \ldots(2.4)$$

$N_2^{=}(v_3) = \alpha-2$ and $N_2^{+}(v_2) = \beta-2$. Thus , by (1.7)

$$C_n(v_3, G, 2) = C_n(v_{\alpha-1}, G, 2) = \binom{p-4}{n-1} - \binom{\beta-2}{n-1}. \quad \ldots(2.5)$$

For $i = 4, 5, \ldots, \alpha-2$, then $N_2^{=}(v_i) = \alpha-4$ and $N_2^{+}(v_i) = \beta$. Thus, by (1.7)

$$C_n(v_i, G, 2) = \binom{p-4}{n-1} - \binom{\beta}{n-1}. \quad \ldots(2.6)$$

From (2.3) - (2.6) we obtain

$$C_n(V, G, 2) = (\alpha-3)\binom{p-4}{n-1} + 2\binom{p-6}{n-1} - (\alpha-6)\binom{\beta}{n-1} - 3\binom{\beta-2}{n-1} - 2\binom{\beta-4}{n-1}. \quad \ldots(2.7)$$

Now, we find $C_n(U, G, 2)$.
$N_2^{=}(u_1) = 5$ and $N_2^{+}(u_1) = p-10$. Thus, by (1.7)

$$C_n(u_1, G, 2) = C_n(u_\beta, G, 2) = \binom{p-5}{n-1} - \binom{p-10}{n-1}. \quad \ldots(2.8)$$

$N_2^{=}(u_2)=4$ and $N_2^{+}(u_2)=p-7$. Thus , by (1.7)

$$C_n(u_2,G,2)=C_n(u_{\beta-1},G,2)=\binom{p-3}{n-1}-\binom{p-7}{n-1}. \quad \ldots(2.9)$$

For u_i , $i=3,4,\ \ldots,\beta-2$, then $N_2^{=}(u_i)=2$ and $N_2^{+}(u_i)=p-5$. Thus , by (1.7)

$$C_n(u_i,G,2)=\binom{p-3}{n-1}-\binom{p-5}{n-1},\ i=3,4,\ \ldots,\beta-2. \quad \ldots(2.10)$$

From (2.8) , (2.9) , and (2.10) we get

$$C_n(U,G,2)=(\beta-2)\binom{p-3}{n-1}-(\beta-6)\binom{p-5}{n-1}-2\binom{p-7}{n-1}-2\binom{p-10}{n-1}. \quad \ldots(2.11)$$

Since $C_n(G,2)=C_n(V,G,2)+C_n(U,G,2)$, then from (2.7) and (2.11), we get (2.2). #

By the method used in proving Proposition 2.2 , we get $C_n(G,3)$.

Proposition 2.3: For $2\le n\le p$ and $\alpha,\beta\ge 7$, we have

$$C_n(G,3)=(\beta-4)\binom{p-5}{n-1}+(\alpha-5)\binom{\beta}{n-1}-(\alpha-8)\binom{\beta-2}{n-1}-\binom{\beta-4}{n-1}-2\binom{\beta-6}{n-1}$$
$$-(\beta-8)\binom{p-7}{n-1}-2\binom{p-9}{n-1}+2\binom{p-10}{n-1}-2\binom{p-12}{n-1}-2\binom{\beta-7}{n-1} \quad \ldots(2.12)$$

Proof: From Fig. 2.1 and by using a procedure that is used in proving Proposition 2.2, we get:

$$C_n(v_1,G,3)=\binom{\beta-2}{n-1}-\binom{\beta-4}{n-1},$$

$$C_n(v_2,G,3)=C_n(v_\alpha,G,3)=\binom{\beta-4}{n-1}-\binom{\beta-6}{n-1}$$

$$C_n(v_3,G,3)=C_n(v_{\alpha-1},G,3)=\binom{\beta-2}{n-1}-\binom{\beta-4}{n-1},$$

$$C_n(v_i,G,3)=\binom{\beta}{n-1}-\binom{\beta-2}{n-1},\ i=4,5,\ \ldots,\alpha-2.$$

Thus,

$$C_n(V,G,3)=(\alpha-5)\binom{\beta}{n-1}-(\alpha-8)\binom{\beta-2}{n-1}-\binom{\beta-4}{n-1}-2\binom{\beta-6}{n-1}. \quad \ldots(2.13)$$

Moreover,

$$C_n(u_1,G,3)=C_n(u_\beta,G,3)=\binom{p-10}{n-1}-\binom{\beta-7}{n-1},$$

$$C_n(u_2,G,3)=C_n(u_{\beta-1},G,3)=\binom{p-7}{n-1}-\binom{p-12}{n-1},$$

$$C_n(u_3,G,3)=C_n(u_{\beta-2},G,3)=\binom{p-5}{n-1}-\binom{p-9}{n-1},$$

$$C_n(u_i,G,3)=\binom{p-5}{n-1}-\binom{p-7}{n-1},\ i=4,5,\ \ldots,\ \beta-3.$$

Thus,

$$C_n(U,G,3)=(\beta-4)\binom{p-5}{n-1}-(\beta-8)\binom{p-7}{n-1}-2\binom{p-12}{n-1}-2\binom{\beta-7}{n-1}+2\binom{p-10}{n-1}-2\binom{p-9}{n-1}. \quad \ldots(2.14)$$

Since $C_n(G,3)=C_n(V,G,3)+C_n(U,G,3)$, then from (2.13) and (2.14), we obtain (2.12). #

We shall obtain $C_n(V,G,k)$ and $C_n(U,G,k)$, for $4\le k\le \mathrm{diam}_n(G)=\delta_n$. We assume that $3\le n\le p$, is the order of G, see Fig, 2.1.
It is clear that
$N_k^{=}(v_2)=2$ and $N_k^{+}(v_2)=\beta-2k$, for $4\le k\le \delta_n-2$, and
$N_k^{=}(v_2)=N_k^{+}(v_2)=0$, for $k=\delta_n-1$ and $\delta_n\ge 6$.
Thus,

$$C_n(v_2,G,k)=\binom{\beta-2k+2}{n-1}-\binom{\beta-2k}{n-1}=C_n(v_\alpha,G,k),\ \text{for } 4\le k\le \delta_n. \quad \ldots(2.15)$$

Similarly, for $4\le k\le \delta_n-1$, $(\delta_n\ge 5)$
$N_k^{=}(v_1)=2$ and $N_k^{+}(v_1)=\beta-2k+2$,
$N_{\delta_n}^{=}(v_1)=N_{\delta_n}^{+}(v_1)=0$, therefore, for $4\le k\le \delta_n$, we have

$$C_n(v_1,G,k)=C_n(v_3,G,k)=C_n(v_{\alpha-1},G,k)=\binom{\beta-2k+4}{n-1}-\binom{\beta-2k+2}{n-1}. \quad \ldots(2.16)$$

Finally, for $4\le k\le \delta_n$ and $i=4,5,\ \ldots,\ \alpha-2$
$N_k^{=}(v_i)=2$ and $N_k^{+}(v_i)=\beta-2k+4$, thus for $4\le k\le \delta_n$, we have

$$C_n(v_i,G,k)=\binom{\beta-2k+6}{n-1}-\binom{\beta-2k+4}{n-1},\ \text{for } i=4,5,\ \ldots,\ \alpha-2. \quad \ldots(2.17)$$

Hence, from (2.15), (2.16) and (2.17), we have

Proposition 2.4: For $4\le k\le \delta_n$ and $3\le n\le p$,

$$C_n(V,G,k)=(\alpha-5)\binom{\beta-2k+6}{n-1}-(\alpha-8)\binom{\beta-2k+4}{n-1}-\binom{\beta-2k+2}{n-1}-2\binom{\beta-2k}{n-1} \quad \ldots(2.18)$$

Remark (1): It is clear from Proposition 2.1 that for all values of n,

$$\delta_n=\mathrm{diam}_n(G)\le\left\lfloor\frac{\beta-1}{2}\right\rfloor+3=\begin{cases}\dfrac{\beta}{2}+2, & \text{for even } \beta,\\ \dfrac{\beta-1}{2}+3, & \text{for odd } \beta.\end{cases}$$

To find $C_n(U,G,k)$, $4\le k\le \delta_n$, we shall consider the main two cases for β, namely even or odd.

(a). Let β be even and denoted $t=\beta/2$:

From Fig.2.1 , we get for $j=1,2,\ \dots,\ k-3$, and $k=4,5,\ \dots,\ t-1$

$N_k^{=}(u_j)=2$ and $N_k^{+}(u_j)=\beta-2k-1$, therefore, for $k=4,5,\ \dots,\ t-1$

$$C_n(u_j,G,k)=C_n(u_{\beta-j+1},G,k)=\binom{\beta-2k+1}{n-1}-\binom{\beta-2k-1}{n-1},\quad j=1,2,\ \dots,\ k-3. \quad \dots(2.19)$$

Also , for $k=4,5,\ \dots,\ t-1$,

$N_k^{=}(u_{k-2})=\alpha-3$ and $N_k^{+}(u_{k-2})=\beta-2k-1$, therefore

$$C_n(u_{k-2},G,k)=C_n(u_{\beta-k+3},G,k)=\binom{p-2k-4}{n-1}-\binom{\beta-2k-1}{n-1},\ \text{for } 4\le k\le t-1. \quad \dots(2.20)$$

Similarly , for the same values of k , namely $4\le k\le t-1$,

$N_k^{=}(u_{k-1})=5$ and $N_k^{+}(u_{k-1})=p-2k-6$, therefore

$$C_n(u_{k-1},G,k)=C_n(u_{\beta-k+2},G,k)=\binom{p-2k-1}{n-1}-\binom{p-2k-6}{n-1},\ \text{for } 4\le k\le t-1. \quad \dots(2.21)$$

Also , $N_k^{=}(u_k)=4$ and $N_k^{+}(u_k)=p-2k-3$, therefore

$$C_n(u_k,G,k)=C_n(u_{\beta-k+1},G,k)=\binom{p-2k+1}{n-1}-\binom{p-2k-3}{n-1},\ \text{for } 4\le k\le t-1. \quad \dots(2.22)$$

Moreover , for $j=k+1,k+2,\ \dots,\ \beta/2\ (=t)$, we have

$N_k^{=}(u_j)=2$ and $N_k^{+}(u_j)=p-2k-1$, for $4\le k\le t-1$.

Therefore , for $4\le k\le t-1$, we get

$$C_n(u_j,G,k)=C_n(u_{\beta-j+1},G,k)=\binom{p-2k+1}{n-1}-\binom{p-2k-1}{n-1},$$
$$\text{for } j=k+1,k+2,\ \dots,\ \beta/2\ (=t). \quad \dots(2.23)$$

From (2.19) - (2.23) , we obtain the following statement .

Proposition 2.5: For $4\le k\le t-1$ and $3\le n\le p$, we have

$$\begin{aligned} C_n(U,G,k) = {} & 2(k-3)\binom{\beta-2k+1}{n-1}+2\binom{\beta-2k-4}{n-1}+(\beta-2k+2)\binom{p-2k+1}{n-1} \\ & -2(k-2)\binom{\beta-2k-1}{n-1}-(\beta-2k-2)\binom{p-2k-1}{n-1}-2\binom{p-2k-6}{n-1} \\ & -2\binom{p-2k-3}{n-1}.\ \# \end{aligned} \quad \dots(2.24)$$

For the other values of k , we have the following result :

Proposition 2.6: For $k=t,t+1$ and $t+2$, $3\le n\le p$, we have:

$$\text{(I). } C_n(U,G,t)=2\left[\binom{\alpha+1}{n-1}+\binom{\alpha-1}{n-1}+\binom{\alpha-4}{n-1}-\binom{\alpha-2}{n-1}-\binom{\alpha-5}{n-1}\right]. \quad \dots(2.25)$$

$$\text{(II). } C_n(U,G,t+1)=2\binom{\alpha-2}{n-1}. \quad \dots(2.26)$$

(III). $C_n(U,G,t+2) = 2\binom{\alpha-5}{n-1}$. …(2.27)

Proof : (I) For $k = t$, we have :

$N_t^{=}(u_i) = 1$ and $N_t^{+}(u_i) = 0$, for $i = 1,2, \ldots, t-3$, thus

$$C_n(u_i,G,t) = C_n(u_{\beta-i+1},G,t) = 0, \text{ for } i = 1,2, \ldots, t-3. \quad \ldots(2.25.a)$$

$N_t^{=}(u_{t-2}) = \alpha - 4$ and $N_t^{+}(u_{t-2}) = 0$, thus

$$C_n(u_{t-2},G,t) = C_n(u_{\beta-t+3},G,t) = \binom{\alpha-4}{n-1}. \quad \ldots(2.25.b)$$

$N_t^{=}(u_{t-1}) = 4$ and $N_t^{+}(u_{t-1}) = \alpha - 5$, thus

$$C_n(u_{t-1},G,t) = C_n(u_{\beta-t+2},G,t) = \binom{\alpha-1}{n-1} - \binom{\alpha-5}{n-1}. \quad \ldots(2.25.c)$$

$N_t^{=}(u_t) = 3$ and $N_t^{+}(u_t) = \alpha - 2$, thus

$$C_n(u_t,G,t) = C_n(u_{\beta-t+1},G,t) = \binom{\alpha+1}{n-1} - \binom{\alpha-2}{n-1}. \quad \ldots(2.25.d)$$

Hence , from (2.25.a) - (2.25.d) , we obtain (2.25) .

(II) For $k = t+1$, we have :

$N_{t+1}^{=}(u_i) = N_{t+1}^{+}(u_i) = 0$, for $i = 1,2, \ldots, t-2$.

$$C_n(u_i,G,t+1) = C_n(u_{\beta-i+1},G,t+1) = 0, \text{ for } i = 1,2, \ldots, t-2. \quad \ldots(2.26.a)$$

$N_{t+1}^{=}(u_{t-1}) = \alpha - 5$ and $N_{t+1}^{+}(u_{t-1}) = 0$, thus

$$C_n(u_{t-1},G,t+1) = C_n(u_{\beta-t+2},G,t+1) = \binom{\alpha-5}{n-1}. \quad \ldots(2.26.b)$$

$N_{t+1}^{=}(u_t) = 3$ and $N_{t+1}^{+}(u_t) = \alpha - 5$, thus

$$C_n(u_t,G,t+1) = C_n(u_{\beta-t+1},G,t+1) = \binom{\alpha-2}{n-1} - \binom{\alpha-5}{n-1}. \quad \ldots(2.26.c)$$

Hence , from (2.26.a) , (2.26.b) and (2.26.c) we obtain (2.26) .

(III) For $k = t+2$, we have :

$N_{t+2}^{=}(u_i) = N_{t+2}^{+}(u_i) = 0$, for $i = 1,2, \ldots, t-1$. …(2.27.a)

$N_{t+2}^{=}(u_t) = \alpha - 5$ and $N_{t+2}^{+}(u_t) = 0$, thus

$$C_n(u_t,G,t+2) = \binom{\alpha-5}{n-1}. \quad \ldots(2.27.b)$$

Hence , from (2.27.a) and (2.27.b), we obtain (2.27) . This completes the proof . #

(b). Let β be odd :

Remark (2): One may check that Proposition 2.5 holds for **odd** β and for $4 \le k \le \frac{\beta-1}{2}$, that is $C_n(U,G,k)$ is given in (2.24) for odd β and $4 \le k \le \frac{\beta-1}{2}$.

Let $t' = \frac{\beta-1}{2}$, where β is odd , $\beta \geq 9$.

Proposition 2.7: For odd β and $k = t'+1, t'+2$ and $t'+3$, $3 \leq n \leq p$, we have:

(I'). $C_n(U,G,t'+1) = \binom{\alpha}{n-1} + \binom{\alpha-2}{n-1}$. …(2.28)

(II'). $C_n(U,G,t'+2) = \binom{\alpha-2}{n-1} + \binom{\alpha-5}{n-1}$. …(2.29)

(III'). $C_n(U,G,t'+3) = \binom{\alpha-5}{n-1}$. …(2.30)

Proof: For each vertex u_j , there is no vertex of U which is of distance $t'+1$, $t'+2$ or $t'+3$ from u_j. Therefore , the only vertices of distance $t'+1$, $t'+2$ or $t'+3$ from each u_j are in W_α. Moreover , the vertices of U that are of distance $t'+1$, $t'+2$ or $t'+3$ from vertices of W_α are $u_{t'-1}$, $u_{t'}$ and $u_{t'+1}$, (and by symmetry $u_{t'+2}$ and $u_{t'+3}$).

(I'). For $k = t'+1$, we have

$N^{=}_{t'+1}(u_{t'-1}) = \alpha - 5$ and $N^{+}_{t'+1}(u_{t'-1}) = 0$, thus

$C_n(u_{t'-1},G,t'+1) = C_n(u_{t'+3},G,t'+1) = \binom{\alpha-5}{n-1}$. …(2.28.a)

$N^{=}_{t'+1}(u_{t'}) = 3$ and $N^{+}_{t'+1}(u_{t'}) = \alpha - 5$, thus

$C_n(u_{t'},G,t'+1) = C_n(u_{t'+2},G,t'+1) = \binom{\alpha-2}{n-1} - \binom{\alpha-5}{n-1}$. …(2.28.b)

$N^{=}_{t'+1}(u_{t'+1}) = 2$ and $N^{+}_{t'+1}(u_{t'+1}) = \alpha - 2$, thus

$C_n(u_{t'+1},G,t'+1) = \binom{\alpha}{n-1} - \binom{\alpha-2}{n-1}$. …(2.28.c)

Hence , from (2.28.a) , (2.28.b) and (2.28.c) we obtain (2.28) .

(II'). For $k = t'+2$, we have

$N^{=}_{t'+2}(u_{t'-1}) = N^{+}_{t'+2}(u_{t'-1}) = 0$, thus

$C_n(u_{t'-1},G,t'+2) = C_n(u_{t'+3},G,t'+2) = 0$. …(2.29.a)

$N^{=}_{t'+2}(u_{t'}) = \alpha - 5$ and $N^{+}_{t'+2}(u_{t'}) = 0$, thus

$C_n(u_{t'},G,t'+2) = C_n(u_{t'+2},G,t'+2) = \binom{\alpha-5}{n-1}$. …(2.28.b)

$N^{=}_{t'+2}(u_{t'+1}) = 3$ and $N^{+}_{t'+2}(u_{t'+1}) = \alpha - 5$, thus

$C_n(u_{t'+1},G,t'+2) = \binom{\alpha-2}{n-1} - \binom{\alpha-5}{n-1}$. …(2.29.c)

Hence , from (2.29.a) , (2.29.b) and (2.29.c) we obtain (2.29).

(III'). For $k = t'+3$, we have

$N^{=}_{t'+3}(u_{t'-1}) = N^{+}_{t'+3}(u_{t'-1}) = 0$, thus

$C_n(u_{t'-1}, G, t'+3) = C_n(u_{t'+3}, G, t'+3) = 0$. …(2.30.a)

$N^{=}_{t'+3}(u_{t'}) = N^{+}_{t'+3}(u_{t'}) = 0$, thus

$C_n(u_{t'}, G, t'+3) = C_n(u_{t'+2}, G, t'+3) = 0$. …(2.30.b)

$N^{=}_{t'+3}(u_{t'+1}) = \alpha - 5$ and $N^{+}_{t'+3}(u_{t'+1}) = 0$, thus

$$C_n(u_{t'+1}, G, t'+3) = \binom{\alpha-5}{n-1}. \qquad \text{…(2.30.c)}$$

Hence , from (2.30.a) , (2.30.b) and (2.30.c) we obtain (2.30).
This completes the proof . #

From (2.18) and (2.24), we obtain $C_n(G,k)$ for $4 \le k \le t-1$ for **even** β as given next :

$C_n(G,k) = C_n(U,G,k) + C_n(V,G,k)$, $4 \le k \le t-1$, even β,

$$= (\alpha-5)\binom{\beta-2k+6}{n-1} + 2(k-3)\binom{\beta-2k+1}{n-1} + 2\binom{p-2k-4}{n-1} + (\beta-2k+2)\binom{p-2k+1}{n-1}$$
$$-(\alpha-8)\binom{\beta-2k+4}{n-1} - \binom{\beta-2k+2}{n-1} - 2\binom{\beta-2k}{n-1} - 2(k-2)\binom{\beta-2k-1}{n-1}$$
$$-(\beta-2k-2)\binom{p-2k-1}{n-1} - 2\binom{p-2k-6}{n-1} - 2\binom{p-2k-3}{n-1}.$$

For other values of k, we have from (2.18) and (2.25)-(2.28):

$$C_n(G,t) = 2\left[\binom{\alpha+1}{n-1} + \binom{\alpha-1}{n-1} + \binom{\alpha-4}{n-1} - \binom{\alpha-2}{n-1} - \binom{\alpha-5}{n-1}\right] + (\alpha-5)\binom{6}{n-1}$$
$$-(\alpha-8)\binom{4}{n-1} - \binom{2}{n-1}. \qquad \text{…(2.32)}$$

$$C_n(G,t+1) = 2\binom{\alpha-2}{n-1} + (\alpha-5)\binom{4}{n-1} - (\alpha-8)\binom{2}{n-1}. \qquad \text{…(2.33)}$$

$$C_n(G,t+2) = 2\binom{\alpha-5}{n-1} + (\alpha-5)\binom{2}{n-1}. \qquad \text{…(2.34)}$$

The formula (2.31) holds also for **odd** β and $4 \le k \le t' = \frac{\beta-1}{2}$. This is required to find $C_n(G,k)$ for other values of k , namely $t'+1$, $t'+2$ and $t'+3$.

Hence, for **odd** β :

$$C_n(G,t'+1) = (\alpha-5)\binom{6}{n-1} - (\alpha-8)\binom{4}{n-1} - 2\binom{2}{n-1} + \binom{\alpha}{n-1} + \binom{\alpha-2}{n-1} \qquad \text{...(2.35)}$$

$$C_n(G,t'+2) = (\alpha-5)\binom{3}{n-1} + \binom{\alpha-2}{n-1} + \binom{\alpha-5}{n-1}. \qquad \text{…(2.36)}$$

$$C_n(G,t'+3) = \binom{\alpha-5}{n-1}. \qquad \text{…(2.37)}$$

Now, we can state the main theorem :

Theorem 2.8: For $3 \le n \le p(=\alpha+\beta)$, $\alpha \ge 8$, $\beta \ge 10$ we have:

$H_n(G;x) = \sum_{k=1}^{\delta_n} C_n(G,k)x^k$, and $W_n(G) = \sum_{k=1}^{\delta_n} k\,C_n(G,k)$

where δ_n is the n-diameter determined by Proposition 2.1, and $C_n(G,k)$ is given in (2.31) for $4 \le k \le \frac{\beta}{2}-1$, for even β; ($4 \le k \le \frac{\beta-1}{2}$, for odd β), and for $k = \frac{\beta}{2}, \frac{\beta}{2}+1$, and $\frac{\beta}{2}+2$, $C_n(G,k)$ is given in (2.32) , (2.33) and (2.34), respectively, for even β; but for $k = \frac{\beta+1}{2}$, $\frac{\beta+3}{2}$, and $\frac{\beta+5}{2}$, $C_n(G,k)$ is given by (2.35), (2.36) and (2.37) for odd β. For $k = 1, 2$ and 3, $C_n(G,k)$ is given by (2.1), (2.2) and (2.3), respectively . #

Remark (3): For $4 \le \alpha \le 7$ and $4 \le \beta \le 9$, one can easily find $H_n(G;x)$ and $W_n(G)$ by direct calculation of $C_n(G,k)$, $1 \le k \le 7$.

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