Thin Films Flow Driven on an Inclined Surface

Khider M.S. Khider
College of Computer Sciences and Mathematics
University of Mosul

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الملخص

يهدف البحث إلى دراسة ميكانيكية الجريان اللامستقر واللامنضغط للأغشية الرقيقة بانعدام قوى القصور الذاتي وفي نظام ثنائي البعد، وقد استخدمت معادلة الاستمرارية ومعادلات نافير-ستوكس لإيجاد المعادلة التي تحكم هذا النوع من الجريان.

ABSTRACT

The flow of unsteady incompressible two dimensional system flow of a thin liquid films with negligible inertia is investigated. Continuity equation and Navier-Stokes equations are used to obtain the equation that governs this type of flow.

Introduction:

The flow of thin films of fluids is encountered in many engineering and biological applications. They include; the flow of rainwater on a road, windscreen or other draining problem [3], paint and coating flow [6, 1]. The flow of many protective biological fluids [5], and other coating are paint and dry processes [4, 7, 8]. The fluid film thickness and the average fluid flux are the main characteristics of interest in these applications [9]. Bascom, Cottington, and Singleterry [10] reported experimental observations of contact lines of thin liquid films. Emilia Borsa had studied the flow of a thin layer on a horizontal plate in the lubrication approximation[2]. The objective here is to obtain the equation governing the flow in thin liquid films, and to find the thickness of the film.

Governing Equations:

We consider a two-dimensional thin film flow on an inclined plane at angle α. The x-axis is oriented stream wise along the plane. The y-axis is perpendicular to the plane in the film thickness direction with the origin at the liquid plane interface. The flow is considered to be a laminar incompressible Newtonian fluid with constant density ρ and constant viscosity μ, and governed by the Navier-Stokes equations and continuity equation as:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho g_x \qquad \ldots(1)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\rho g_y \qquad \ldots(2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \ldots(3)$$

In the thin film lubrication approximation the Navier-Stokes equations read

$$0=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\rho g\sin(\alpha) \qquad \ldots(4)$$

$$0=-\frac{\partial p}{\partial y}-\rho g\cos(\alpha) \qquad \ldots(5)$$

Where g is gravitational acceleration, y=h(x,y) is the free surface, p is the pressure in the fluid , u and p depended on X=(x,y,t), and t is the time.

To complete the problem formulation, the lubrication equations (3), (4) and (5) require the boundary conditions:

$$u=0 \quad on \quad y=0 \qquad \ldots(6)$$

$$\tau=\mu\frac{\partial u}{\partial y} \quad on \quad y=h \qquad \ldots(7)$$

$$p=p_0 \quad on \quad y=h \qquad \ldots(8)$$

Where p_0 is the atmospheric pressure in the air face.

$$v=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} \quad on \quad y=h \qquad \ldots(9)$$

Where the boundary conditions (6), (7), (8) and (9) represent the no-slip condition ,the balance of tangential stress, the balance of normal stress and the kinematics condition respectively.

Integrating (4) with respect to y and using the boundary condition (7) we have:

$$\mu\frac{\partial u}{\partial y}=\frac{\partial p}{\partial x}y-\rho g\sin(\alpha)y+\tau-\frac{\partial p}{\partial x}h+\rho g\sin(\alpha)h \qquad \ldots(10)$$

Similarly integrating (10) with respect to y and using boundary condition (6) we get :

$$u=\frac{1}{2\mu}\frac{\partial p}{\partial x}y^2+\frac{\tau}{\mu}y-\frac{\rho g}{2\mu}\sin(\alpha)y^2-\frac{1}{\mu}\frac{\partial p}{\partial x}hy+\frac{\rho g}{\mu}\sin(\alpha)hy \qquad \ldots(11)$$

Now, integrating (5) with respect to y and using the boundary condition (8) we obtain :

$$p=-\rho g\cos(\alpha)y+p_0+\rho g\cos(\alpha)h \qquad \ldots(12)$$

Drive(12) with respect to x and substitution in to(11) we have:

$$u=\frac{\rho g}{2\mu}\cos(\alpha)\frac{\partial h}{\partial x}y^2-\frac{\rho g}{\mu}\sin(\alpha)y^2+\left(\frac{\tau}{\mu}-\frac{\rho g}{\mu}\cos(\alpha)\frac{\partial h}{\partial x}h+\frac{\rho g}{\mu}\sin(\alpha)h\right)y \qquad \ldots(13)$$

By using the continuity equation in the thin film approximation and the kinematics boundary condition leads the evolution equation for $y=h(x,t)$

$$\frac{\rho g}{2\mu}\cos(\alpha)\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right)-\frac{3\rho g}{2\mu}\sin(\alpha)\frac{\partial h}{\partial x}h^2=\frac{\partial h}{\partial t}+\frac{\tau}{\mu}\frac{\partial h}{\partial x}h \qquad \ldots(14)$$

We introduce the following nondimensional variables defined by:

$$\bar{h}=\frac{h}{h_0}\ ,\ \bar{x}=\frac{x}{L}\ ,\ \bar{t}=\frac{tU}{h_0}\ ,\ \bar{\tau}=\frac{L\tau}{\sigma} \qquad \ldots(15)$$

Where the velocity U and the length scale L are characteristic quantities of the problem, assume that δ=ho/L, where ho is the characteristic length for the film thickness.

Then, convert the equation (14) into no dimensional form in terms of the no dimensional variables h^-, x^-, t^-, τ^- and the equation (14) becomes

$$Bo\cos(\alpha)\frac{\partial}{\partial\bar{x}}\left(\bar{h}^3\frac{\partial\bar{h}}{\partial\bar{x}}\right)=\frac{\partial\bar{h}}{\partial\bar{t}}+\frac{3}{2}\sin(\alpha)k\bar{h}^2\frac{\partial\bar{h}}{\partial\bar{x}}+Ca\,\bar{\tau}\,\bar{h}\frac{\partial\bar{h}}{\partial\bar{x}} \qquad \ldots(16)$$

Where $Bo=\delta^4\frac{\rho g l^2}{3\mu U}$, $Ca=\delta^2\frac{\sigma_0}{\mu U}$, $k=\delta\frac{\rho g {h_0}^2}{\mu U}$

For the sake of simpler notation, we drop the "dash" from the non dimensional variables $\bar{h}$, $\bar{x}$, $\bar{t}$, $\bar{\tau}$ in the equation (16) and we take the unsteady flow the equation (16) and divided on $Bo\cos(\alpha)$, we get:

$$\frac{d}{dx}\left(h^3\frac{\partial h}{\partial x}\right)=\frac{3}{2}H\tan(\alpha)h^2\frac{\partial h}{\partial x}+M\sec(\alpha)\tau\,h\frac{\partial h}{\partial x} \qquad \ldots(17)$$

where $H=\frac{k}{Bo}$, $M=\frac{Ca}{Bo}$

Integrating (17) with respect to x and divide on h^3 we obtain:

$$\frac{\partial h}{\partial x}=\frac{3}{2}H\tan(\alpha)+M\sec(\alpha)\tau\frac{1}{h}+\frac{A}{h^3} \qquad \ldots(18)$$

Where A is constant.

Now, we take two cases:

The first case when $A=0$ *and* $\tau=0$ the equation (18) becomes

$$\frac{\partial h}{\partial x}=\frac{3}{2}H\tan(\alpha) \qquad \ldots(19)a$$

Integrating (19)a with respect to x we have:

$$h(x)=\frac{3}{2}H\tan(\alpha)x+f \qquad \ldots(20)a$$

f is constant.

By giving different value to the constant f, and angle α, we get the thick the film.

The second case when A≠0 and τ=0, the equation (18) becomes:

$$\frac{\partial h}{\partial x}=\frac{3}{2}H\tan(\alpha)+\frac{A}{h^3} \qquad \ldots(19)b$$

Now, to get the initial condition for the equation (19)b , we suppose that $\frac{\partial h}{\partial x}=0$ and the equation (19)b becomes:.

$$h^3(x)=\frac{-A}{\frac{3}{2}H\tan(\alpha)} \qquad \ldots(20)b$$

By taking cubic root the equation (26)b, we have:

$$h(x)=\sqrt[3]{\frac{-A}{\frac{3}{2}H\tan(\alpha)}} \qquad \ldots(21)$$

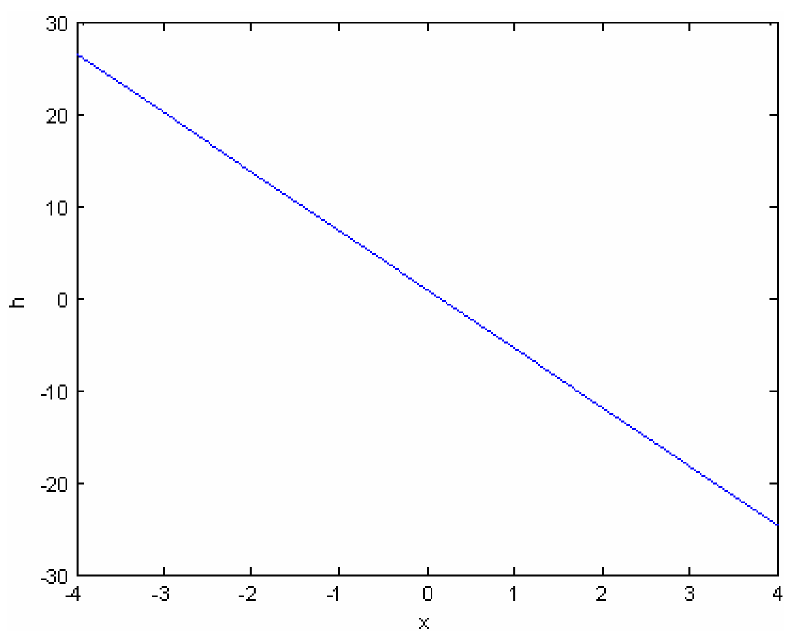
Similarly by taking different value for A and α we get the thick film in another case.

Table (1.1). Represent Solutions of Equation (20)a for Different α

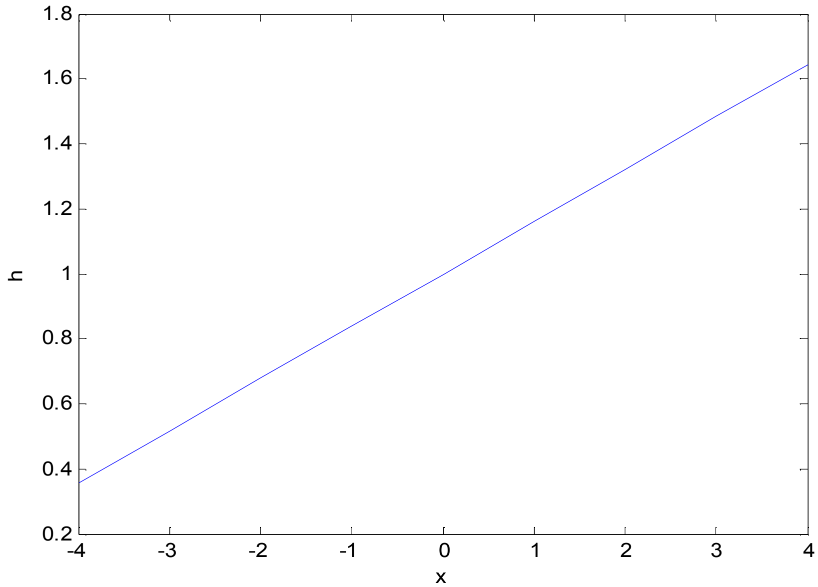
x	$\alpha = 30$	$\alpha = 41$	$\alpha = 49$	$\alpha = 60$
4.0	−24.6213	1.6426	−11.6916	2.2802
3.50	−21.4187	1.5623	−10.1052	2.1201
3.00	−18.2160	1.4820	−8.5187	1.9601
2.50	−15.0133	1.4016	−6.9323	1.8001
2.00	−11.8107	1.3213	−5.3458	1.6401
1.50	−8.6080	1.2410	−3.7594	1.4801
1.00	−5.4053	1.1607	−2.1729	1.3200
0.50	−2.2027	1.0803	−0.5865	1.1600
0	1.0000	1.0000	1.0000	1.0000
−0.50	4.2027	0.9197	2.5865	0.8400
−1.00	7.4053	0.8393	4.1729	0.6800
−1.50	10.6080	0.7590	5.7594	0.5199
−2.00	13.8107	0.6787	7.3458	0.3599
−2.50	17.0133	0.5984	8.9323	0.1999
−3.00	20.2160	0.5180	10.5187	0.0399
−3.50	23.4187	0.4377	12.1052	−0.1201
−4.00	26.6213	0.3574	13.6916	−0.2802

Table (1.2). Represent Solutions of Equation (21) for Different α

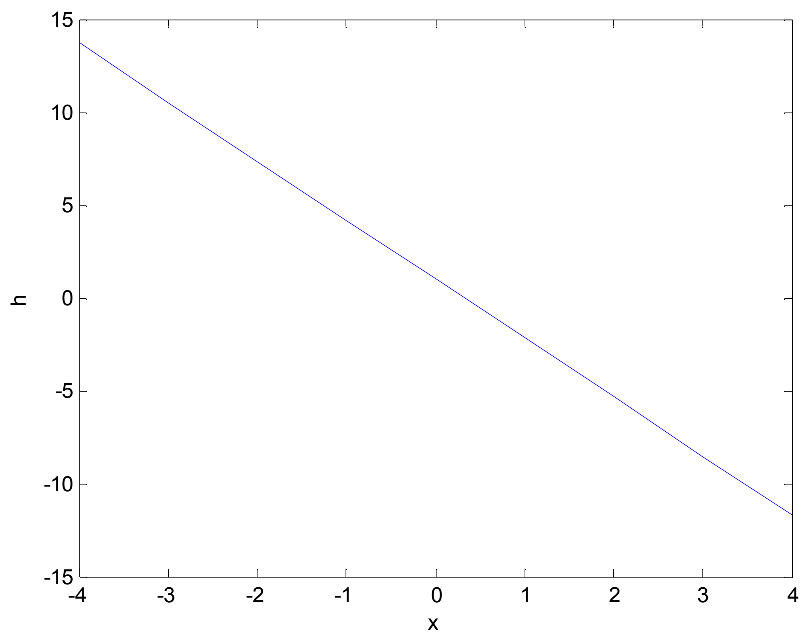
α	30	40	50	60
f	0.2137	0.3825	0.6126	1.1603



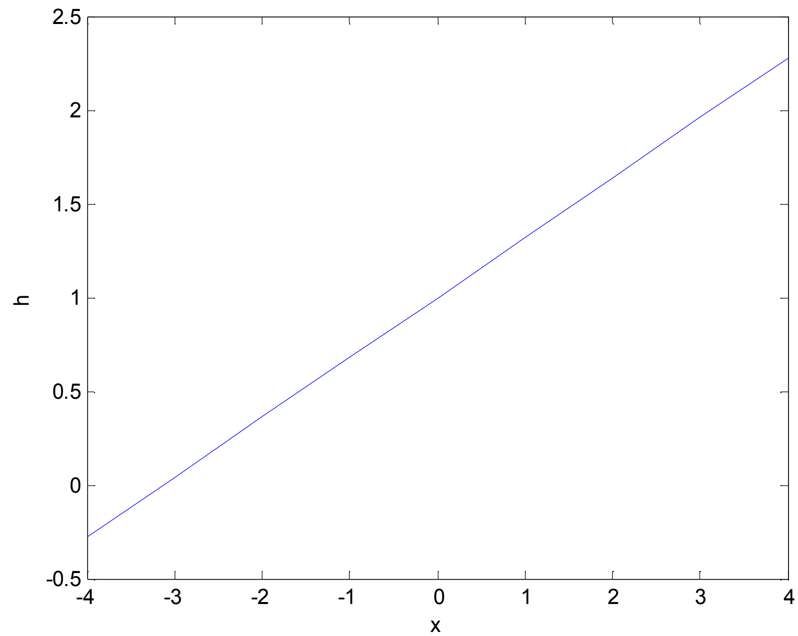
Fig(1.1). Represent Solutions of Equation (20)a f=1, $\alpha = 30$



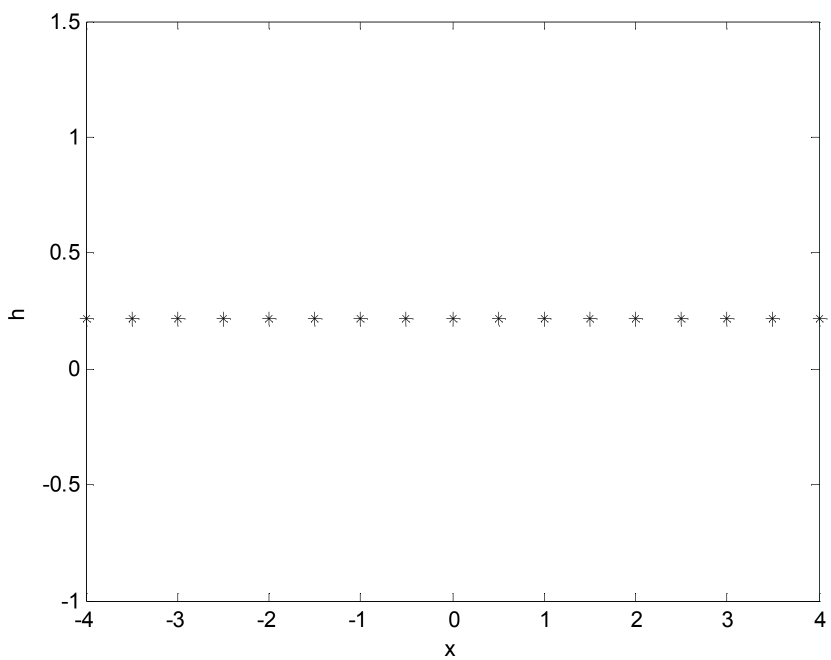
Fig(1.2). Represent Solutions of Equation (20)a f=1, $\alpha = 41$



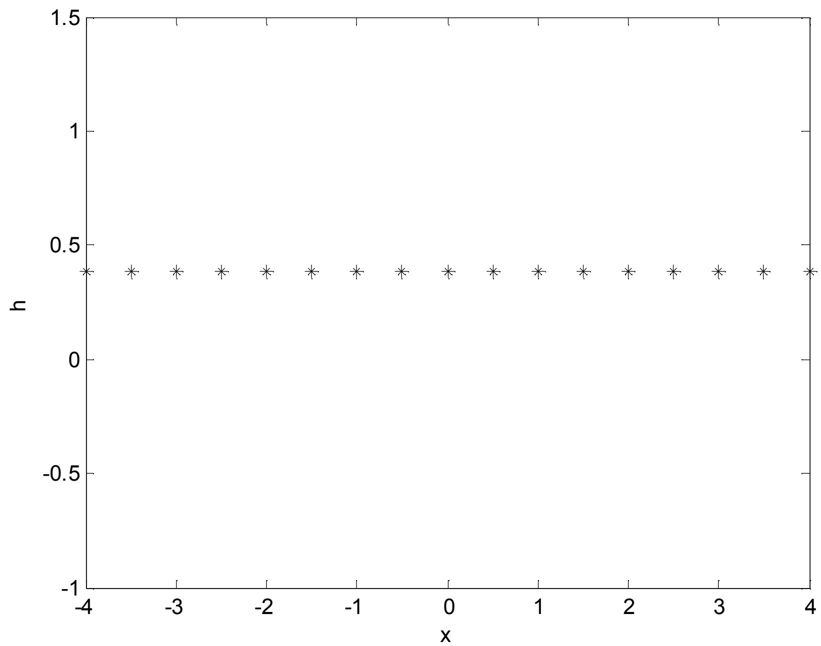
Fig(1.3). Represent Solutions of Equation (20)a f=1, $\alpha = 49$



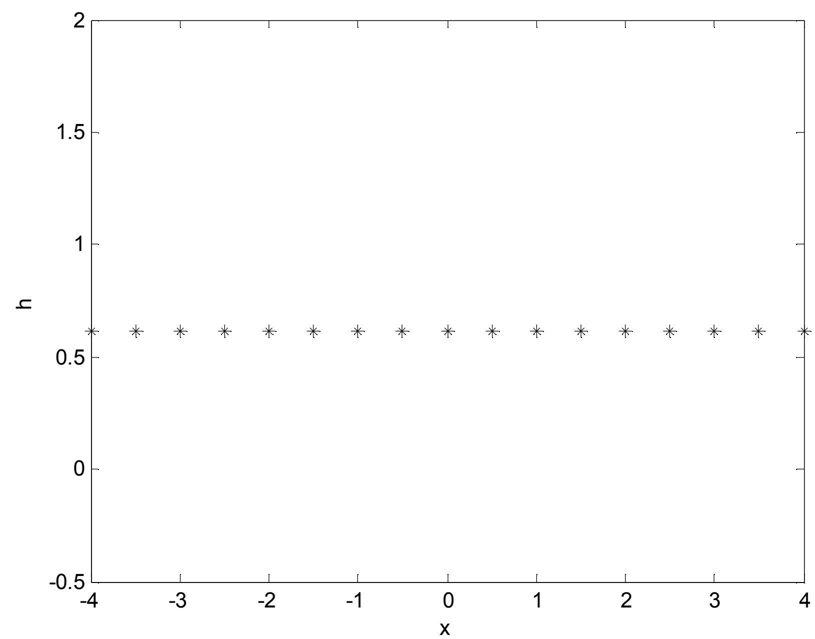
Fig(1.4). Represent Solutions of Equation (20)a f=1, $\alpha = 60$



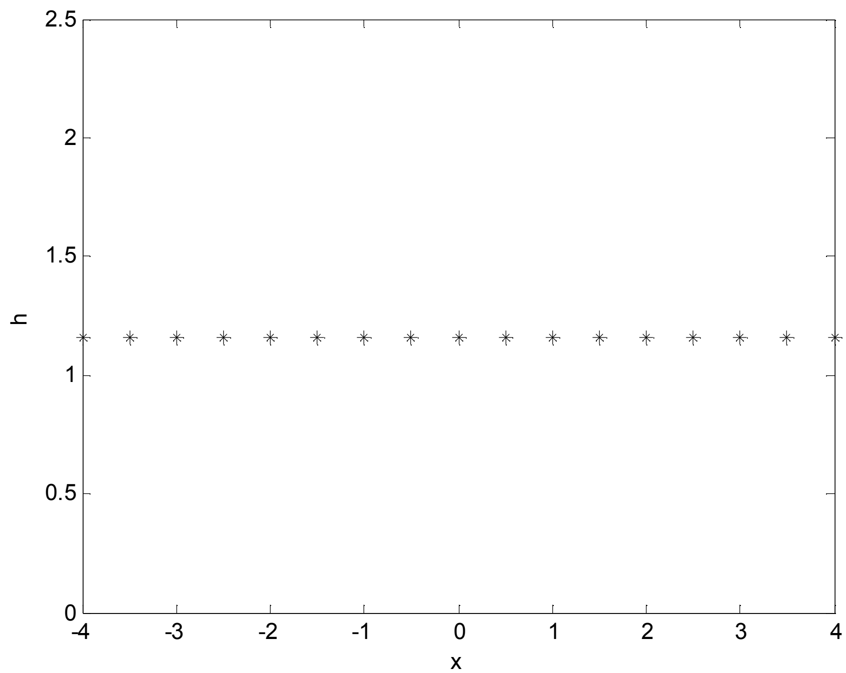
Fig(1.5). Represent Solutions of Equation (21) $\alpha = 30$



Fig(1.6). Represent Solutions of Equation (21) $\alpha = 40$



Fig(1.7). Represent Solutions of Equation (21) $\alpha = 50$



Fig(1.8). Represent Solutions of Equation (21) $\alpha = 60$

Conclusion:

Through our studies to the motion equation for viscous incompressible liquid, we conclude from equation (20)a that the thickness of the film increases when we approach the negative values of x when α=30,49 as shown in Figures (1.1) and (1.3) and it will decrease towards the positive values of x, when α=41, 60 as shown in Figures (1.2) and (1.4) and the table (1.1), this implies that the value of the angle α will affect the thickness of the film. From equation (21), we note that the thickness of the film will be parallel to the x-axis and it will increase according to the value of α as shown in Figures (1.5), (1.6), (1.7) and (1.8) and the table (1.2).   .

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