On iα - Open Sets

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الملخص

في هذا البحث، قدمنا نوعا جديدا من المجاميع المفتوحة المعرفة بالصيغة التالية: المجموعة الجزئية **A** من الفضاء التبولوجي (X,τ) يقال عنها مجموعة مفتوحة من النوع-iα، إذا وجدت مجموعة جزئية فعلية غير خالية O، $O \in \alpha O(X)$ بحيث أن $A \subseteq Cl(A \cap O)$. كذلك قدمنا فكرة التطبيق المستمر من النوع-iα، والتطبيق المفتوح من النوع-iα، والتطبيق المتردد من النوع-iα، و التطبيق الكلي المستمر من النوع-iα ، والتطبيق ضد-المستمر من النوع -i ، والتطبيق ضد-المستمر من النوع –iα مع تحقيق بعض الخصائص لتلك التطبيقات. بالإضافة إلى ذلك، قدمنا بعض بديهيات الفصل من النوع-iα والتطبيقات المرتبطة مع بديهيات الفصل من النوع-iα.

ABSTRACT

In this paper, we introduce a new class of open sets defined as follows: A subset A of a topological space (X,τ) is called iα-open set, if there exists a non-empty subset O of X, $O \in \alpha O(X)$, such that $A \subseteq Cl(A \cap O)$. Also, we present the notion of iα-continuous mapping, iα-open mapping, iα-irresolute mapping, iα-totally continuous mapping, i-contra-continuous mapping, iα-contra-continuous mapping and we investigate some properties of these mappings. Furthermore, we introduce some iα-separation axioms and the mappings are related with iα-separation axioms.

1 Introduction and Preliminaries

A Generalization of the concept of open sets is now well-known important notions in topology and its applications. Levine [7] introduced semi-open set and semi-continuous function, Njastad [8] introduced α-open set, Askander [15] introduced i-open set, i-irresolute mapping and i-homeomorphism, Biswas [6] introduced semi-open functions, Mashhour, Hasanein, and El-Deeb [1] introduced α-continuous and α-open mappings, Noiri [16] introduced totally (perfectly) continuous function, Crossley [11] introduced irresolute function, Maheshwari [14] introduced α-irresolute mapping, Beceren [17] introduced semi α-irresolute functions, Donchev [4] introduced contra continuous functions, Donchev and Noiri [5] introduced contra semi continuous functions, Jafari and Noiri [12] introduced Contra-α-continuous functions, Ekici and Caldas [3] introduced clopen-T1, Staum [10] introduced, ultra hausdorff, ultra normal, clopen regular and clopen normal, Ellis [9] introduced ultra regular, Maheshwari [13] introduced s-normal space, Arhangel [2] introduced α-normal space. The main aim of this paper is to introduce and study a new class of open sets which is called iα-open set and we present the notion of iα-continuous mapping, iα-totally continuity mapping and some weak separation axioms for iα-open sets. Furthermore, we investigate some properties of these mappings. In section 2, we define iα-open set, and we investigate the relationship with, open set, semi-open set, α-open set and i-open set. In section 3, we present the notion of iα-continuous mapping, iα-open mapping, iα-irresolute mapping and iα-homeomorphism mapping, and we investigate the relationship between iα-continuous mapping with some types of continuous mappings, the relationship between iα-open mapping, with some types of open mappings and the relationship between iα-irresolute mapping with some types of irresolute mappings. Further, we compare iα-homeomorphism with i-homeomorphism. In section 4, we introduce new class of mappings called iα-totally continuous mapping and we introduce i-contra-continuous mapping and iα-contra-continuous mapping. Further, we study some of their basic properties. Finally in section 5, we introduce new weak of separation axioms for iα-open set and we conclude iα-continuous mappings related with iα-separation axioms. Throughout this paper, we denote the topology spaces (X,τ) and (Y,σ) simply by X and Y respectively. We recall the following definitions, notations and characterizations. The closure (resp. interior) of a subset A of a topological space X is denoted by $Cl(A)$ (resp. $Int(A)$).

Definition 1.1 A subset A of a topological space X is said to be
(i) semi-open set, if $\exists O \in \tau$ such that $O \subseteq A \subseteq Cl(O)$ [7]
(ii) α –open set, if $A \subseteq Int(Cl(Int(A)))$ [8]
(iii) i-open set, if $A \subseteq Cl(A \cap O)$, where $\exists O \in \tau$ and $O \neq X, \phi$ [15]
(iv) clopen set, if A is open and closed.
The family of all semi-open (resp. α-open , i-open, clopen) sets of a topological space is denoted by $SO(X)$(resp. $\alpha O(X)$, $iO(X)$, $CO(X)$). The complement of semi-open (resp. α-open, i-open) sets of a topological space X is called semi-closed (resp. α-closed, i-closed) sets.

Definition 1.2 Let X and Y be a topological spaces, a mapping $f : X \rightarrow Y$ is said to be
(i) semi-continuous [7] if the inverse image of every open subset of Y is semi-open set in X.
(ii) α-continuous [1] if the inverse image of every open subset of Y is an α-open set in X.
(iii) i-continuous [15] if the inverse image of every open subset of Y is an i-open set in X.
(iv) totally (perfectly) continuous [16] if the inverse image of every open subset of Y is clopen set in X.
(v) irresolute [11] if the inverse image of every semi-open subset of Y is semi-open subset in X.
(vi) α-irresolute [14] if the inverse image of every α-open subset of Y is an α-open subset in X .
(vii) semi α-irresolute [17] if the inverse image of every α-open subset of Y is semi-open subset in X.
(viii) i-irresolute [15] if the inverse image of every i-open subset of Y is an i-open subset in X.
(ix) contra-continuous [4] if the inverse image of every open subset of Y is closed set in X.
(x) contra semi continuous [5] if the inverse image of every open subset of Y is semi-closed set in X.
(xi) contra α-continuous [12] if the inverse image of every of open subset of Y is an α-closed set in X.
(xii) semi-open [6] if the image of every open set in X is semi-open set in Y.
(xiii) α-open [1] if the image of every open set in X is an α-open set in Y.
(xiv) i-open [15] if the image of every open set in X is an i-open set in Y.

Definition 1.3 Let X and Y be a topology space, a bijective mapping $f : X \rightarrow Y$ is said to be i-homeomorphism [15] if f is an i-continuous and i-open.

Lemma 1.4 Every open set in a topological space is an i-open set [15].

Lemma 1.6 Every semi-open set in a topological space is an i-open set [15].

Lemma 1.8 Every α-open set in a topological space is an i-open set [15].

2 Sets That are iα-Open Sets and Some Relations With Other Important Sets

In this section, we introduce a new class of open sets which is called iα-open set and we investigate the relationship with, open set, semi-open set, α-open set and i-open set.

Definition 2.1 A subset A of the topological space X is said to be iα-open set if there exists a non-empty subset O of X, $O \in \alpha O(X)$, such that $A \subseteq Cl(A \cap O)$. The complement of the iα-open set is called iα-closed. We denote the family of all iα-open sets of a topological space X by i$\alpha O(X)$.

Example 2.2
Let X={a,b,c}, τ={Ø,{b},{b,c},X}, $SO(X)$ =$\alpha O(X)$ ={Ø,{b}, {a,b},{b,c},X}and i$\alpha O(X)$={Ø,{a},{b},{c},{a,b},{a,c},{b,c},X}. Note that $SO(X)$ = $\alpha O(X) \subset$ i$\alpha O(X)$.

Example 2.3
Let X={a,b,c,d}, τ={Ø,{a,d},{b,c},X}=$\alpha O(X)$, i$\alpha O(X)$={Ø,{a},{b},{c},{d},{a,d},{b,c}, X}.

Example 2.4
Let X={a,b,c}, τ={Ø,{a},{b},{a,b},X} i$\alpha O(X)$={Ø,{a},{b},{a,b},{a,c},{b,c},X}.

Lemma 2.5 Every i-open set in any topological space is an iα-open set.
Proof. Let X be any topological space and $A \subseteq X$ be any i-open set. Therefore, $A \subseteq Cl(A \cap O)$, where $\exists O \in \tau$ and $O \neq X, \phi$. Since, every open is an α-open[8], then $\exists O \in \alpha O(X)$. We obtain $A \subseteq Cl(A \cap O)$, where $\exists O \in \alpha O(X)$ and $O \neq X, \phi$. Thus, A is an iα-open set■ The following example shows that iα-open set need not be i-open set

Example 2.6 Let X={1,2,3,4}, τ={Ø,{4},X}, $iO(X)$={Ø,{4},{1,4},{2,4},{3,4},{1,2,4}, {1,3,4},{2,3,4},X} ⊂ i$\alpha O(X)${□,{1},{2},{3},{4},{1,2},{1,3,},{1,4},{2,3},{2,4},{3,4}, {1,2,3},{1,2,4},{1,3,4},{2,3,4},X}.

Remark 2.7
(i) The intersection of iα-open sets is not necessary to be iα-open set as shown in the example 2.4.
(ii) The union of iα-open set is not necessary to be iα-open set as shown in the example 2.3.

3 Mappings That are iα-Continuous and iα-Homeomorphism

In this section, we present the notion of iα-continuous mapping, iα-irresolute mapping and iα-homeomorphism mapping.

Definition 3.1 Let X, Y be a topological spaces, a mapping $f : X \to Y$ is said to be iα-continuous, if the inverse image of every open subset of Y is an iα-open set in X.

Example 3.2 Let X=Y={a,b,c}, τ={Ø,{b},{c},{b,c},X}, iαO(X)={Ø,{b},{c},{a,b}, {a,c},{b,c},X} and σ={Ø,{a,b},X}. Clearly, the identity mapping $f : X \to Y$ is an iα-continuous.

Proposition 3.3 Every i-continuous mapping is an iα-continuous.

Proof. Let $f: X \to Y$ be an i-continuous mapping and V be any open subset in Y. Since, f is an i-continuous, then $f^{-1}(V)$ is an i-open set in X. Since, every i-open set is an iα-open set by lemma 2.5, then $f^{-1}(V)$ is an iα-open set in X. Therefore, f is an iα-continuous ■

Remark 3.4 The following example shows that iα-continuous mapping need not be continuous, semi-continuous, α-continuous and i-continuous mappings.

Example 3.5 Let X={a,b,c} and Y={1,2,3}, τ={Ø,{b},X}, $SO(X)=\alpha O(X)=iO(X)$= {Ø,{b},{a,b},{b,c},X}, i$\alpha O(X)$={ Ø,{a},{b},{c},{a,b},{a,c},{b,c},X}, σ={Ø,{2},Y}. A mapping $f: X \to Y$ is defined by f{a}={2}, f{b}={1}, f{c}={3}. Clearly, f is an iα-continuous, but f is not continuous, f is not semi-continuous, f is not α-continuous and f is not i-continuous because for open subset {2},

f^{1}{2}={a} $\notin \tau$ and f^{-1}{2}={a} $\notin SO(X)=\alpha O(X)=iO(X)$.

Definition 3.6 Let X and Y be a topological space, a mapping $f: X \to Y$ is said to be iα-open, if the image of every open set in X is an iα-open set in Y.

Example 3.7 Let X=Y{a,b,c}, τ={Ø,{b,c},X}, σ={Ø,{a},Y}, and i$\alpha O(Y)$={Ø,{a},{b}, {c},{a,b},{a,c},{b,c},Y}. Clearly, the identity mapping $f: X \to Y$ is an iα-open.

Proposition 3.8 Every i-open mapping is an iα-open.

Proof. Let $f: X \to Y$ be an i-open mapping and V be any open set in X. Since, f is an i-open, then $f(V)$ is an i-open set in Y. Since, every i-open set is an iα-open set by lemma 2.5, then $f(V)$is an iα-open set in Y. Therefore, f is an iα-open ■

Remark 3.9 The following example shows that iα-open mapping need not be open, semi-open, α-open and i-open mappings.

Example3.10 Let Let X=Y={1,2,3}, τ={Ø,{3},X} σ={Ø,{1},Y},$SO(Y)=\alpha O(Y)=iO(Y)$, ={Ø,{1},{1,2},{1,3},Y}, i$\alpha O(Y)$={Ø,{1},{2},{3},{1,2},{1,3},{2,3},Y}. A mapping $f: X \to Y$ is defined by f(1)=2, f(2)=1, f(3)=3. Clearly, f is an iα-open, but f is not open, f is not semi-open, f is not α-open and f is not i-open because for open subset{3}, f^{-1}{3}={3} $\notin \sigma$ and f^{-1}{3}={3} $\notin SO(Y)=\alpha O(Y)=iO(Y)$.

Definition 3.11 Let X and Y be a topological space, a mapping $f: X \to Y$ is said to be iα-irresolute, if the inverse image of every iα-open subset of Y is an iα-open subset in X.

Example 3.12 Let X=Y={a,b,c}, τ={Ø,{b},X},i$\alpha O(X)$={Ø,{a},{b},{c},{a,b},{a,c}, {b,c},X}, σ={Ø,{c},Y} and i$\alpha O(Y)$={Ø,{a},{b},{c},{a,b},{a,c},{b,c},Y}. Clearly, the identity mapping $f: X \to Y$ is an iα-irresolute.

Proposition 3.13 Every i-irresolute mapping is an iα-irresolute.

Proof. Let $f: X \to Y$ be an i-irresolute mapping and V be any iα-open set in Y. Since, f is an i-irresolute, then $f^{-1}(V)$ is an i-open set in X. Hence, iα-open set in X by lemma 2.5. Therefore, f is an iα-irresolute■

Remark 3.14 The following example shows that iα-irresolute mapping need not be irresolute, semi α-irresolute, α-irresolute and i-irresolute mappings.

Example 3.15 Let X=Y={a,b,c}, τ={Ø,{a},X}, $SO(X)=\alpha O(X)=iO(X)$={Ø,{a},{a,b}, {a,c},X}, i$\alpha O(X)$={Ø,{a},{b},{c},{a,b},{a,c},{b,c},X}, σ={Ø,{c},Y}, $SO(Y)=\alpha O(Y)$= $iO(Y)$={Ø,{c},{a,c},{b,c},Y} and i$\alpha O(Y)$={Ø,{a},{b},{c},{a,b},{a,c},{b,c},Y}. Clearly, the identity mapping $f : X \rightarrow Y$ is an iα-irresolute, but *f* is not irresolute, *f* is not α-irresolute, *f* is not semi α-irresolute and *f* is not i-irresolute because for semi-open, α-open and i-open subset {c}, f^{-1} {c}={c} $\notin$ *SO*(*X*)=*αO*(*X*)=*iO*(*X*).

Proposition 3.16 Every iα-irresolute mapping is an iα-continuous.

Proof. Let $f : X \rightarrow Y$ be an iα-irresolute mapping and *V* be any open set in *Y*. Since, every open set is an iα-open set. Since, *f* is an iα-irresolute, then $f^{-1}(V)$ is an iα-open set in *X*. Therefore f is an iα- continuous■ The converse of the above proposition need not be true as shown in the following example

Example 4.17 Let *X*=*Y*={a,b,c}, τ={Ø,{a,b},*X*}, iα*O*(*X*)={Ø,{a},{b},{a,b},{a,c},{b,c} ,*X*}, σ={Ø,{a,c},*Y*} and iα*O*(*Y*)={Ø,{a},{b},{c},{a,b},{a,c},{b,c},*Y*}. Clearly, the identity mapping $f : X \rightarrow Y$ is an iα-continuous, but *f* is not iα-irresolute because for iα-open set {c}, f^{-1} {c}={c} $\notin$ iα*O*(*X*).

Definition 3.18 Let *X* and *Y* be a topological space, a bijective mapping $f : X \rightarrow Y$ is said to be iα-homeomorphism if *f* is an iα-continuous and iα-open.

Theorem 3.19 If $f : X \rightarrow Y$ is an i-homomorphism, then $f : X \rightarrow Y$ is an iα-homomorphism.

Proof. Since, every i-continuous mapping is an iα-continuous by proposition 3.3. Also, since every i-open mapping is an iα-open 3.8. Further, since *f* is bijective. Therefore, *f* is an iα-homomorphism ■ The converse of the above theorem need not be true as shown in the following example

Example 3.20

Let *X*=*Y*={a,b,c}, τ={Ø,{a},*X*}, *iO*(*X*)={Ø,{a},{a,b},{a,c},*X*}, α*O*(*X*)={Ø,{a},{b},{c}, {a,b},{a,c},{b,c},*X*}, σ={Ø,{b},*Y*}, *iO*(*Y*)={Ø,{b},{a,b},{b,c},*Y*} and iα*O*(*Y*)={Ø,{a}, {b},{c},{a,b},{a,c},{b,c},*Y*}.Clearly, the identity mapping $f : X \rightarrow Y$ is an iα-homomorphism, but it is not i-homomorphism because *f* is not i-continuous, since for open subset {b}, f^{-1} {b}={b} $\notin$ *iO*(*X*).

4 Mappings That are iα-Totally Continuous and iα-Contra-Continuous

In this section, we introduce new classes of mappings called iα-totally continuous, i-contra-continuous and iα-contra-continuous.

Definition 4.1 Let *X* and *Y* be a topological space, a mapping $f : X \rightarrow Y$ is said to be iα-totally continuous, if the inverse image of every iα-open subset of *Y* is clopen set in *X*.

Example 4.2 Let *X*=*Y*={a,b,c}, τ={Ø,{a},{b,c},*X*}, σ={Ø,{a},*Y*} and iα*O*(*Y*)={Ø,{a}, {b},{c},{a,b},{a,c},{b,c},*Y*}. The mapping $f : X \rightarrow Y$ is defined by *f*{a}={a}, *f*{b}=*f*{c}=b. Clearly, *f* is an iα-totally continuous mapping.

Theorem 4.3 Every iα-totally continuous mapping is totally continuous.

Proof. Let $f : X \rightarrow Y$ be iα-totally continuous and *V* be any open set in *Y*. Since, every open set is an iα-open set, then *V* is an iα-open set in *Y*. Since, *f* is an iα-totally continuous mapping, then $f^{-1}(V)$ is clopen set in *X*. Therefore, *f* is totally continuous■ The converse of the above theorem need not be true as shown in the following example

Example 4.4 Let *X*=*Y*={a,b,c}, τ={Ø,{a},{b,c},*X*} σ={Ø,{a},*Y*} and iα*O*(*Y*)={Ø,{a}, {b},{c},{a,b},{a,c},{b,c},*Y*}. Clearly, the identity mapping $f : X \rightarrow Y$ is totally continuous, but f is not iα-totally continuous because for iα-open set {a,c}, f^{-1}\{a,c\}={a,c} $\notin CO(X)$.

Theorem 4.5 Every iα-totally continuous mapping is an iα-irresolute.

Proof. Let $f: X \to Y$ be iα-totally continuous and V be an iα-open set in Y. Since, f is an iα-totally continuous mapping, then $f^{-1}(V)$ is clopen set in X, which implies $f^{-1}(V)$ open, it follow $f^{-1}(V)$ iα-open set in X. Therefore, f is an iα-irresolute■ The converse of the above theorem need not be true as shown in the following example

Example 4.6 Let X=Y={1,2,3}, τ={Ø,{2},X}, iα$O(X)$={Ø,{1},{2},{3},{1,2},{1,3},{2},{3},{1,2},{1,3},{2,3},X} σ={Ø,{1,2},Y} and iα$O(Y)$={Ø,{1},{2},{1,2},{1,3},{2,3}, Y}. Clearly, the identity mapping $f: X \to Y$ is an iα-irresolute, but f is not iα-totally continuous because for iα-open subset{1,3}, f^{-1}\{1,3\}={1,3} $\notin CO(X)$.

Theorem 4.7 The composition of two iα-totally continuous mapping is also iα-totally continuous.

Proof. Let $f: X \to Y$ and $g: Y \to Z$ be any two iα-totally continuous. Let V be any iα-open in Z. Since, g is an iα-totally continuous, then $g^{-1}(V)$ is clopen set in Y, which implies $f^{-1}(V)$ open set, it follow $f^{-1}(V)$ iα-open set. Since, f is an iα-totally continuous, then $f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V)$ is clopen in X. Therefore, $gof: X \to Z$ is an iα-totally continuous■

Theorem 4.8 If $f: X \to Y$ be an iα-totally continuous and $g: Y \to Z$ be an iα-irresolute, then $gof: X \to Z$ is an iα-totally continuous.

Proof. Let $f: X \to Y$ be iα-totally continuous and $g: Y \to Z$ be iα-irresolute . Let V be iα-open set in Z. Since, g is an iα-irresolute, then $g^{-1}(V)$ is an iα-open set in Y. Since, f is an iα-totally continuous, then $f^{-1}((g^{-1}(V))=(g \circ f)^{-1}(V)$ is clopen set in X. Therefore, $gof: X \to Z$ is an iα-totally continuous■

Theorem 4.9 If $f: X \to Y$ is an iα-totally continuous and $g: Y \to Z$ is an iα-continuous, then $gof: X \to Z$ is totally continuous.

Proof. Let $f: X \to Y$ be iα-totally continuous and $g: Y \to Z$ is an iα-continuous . Let V be an open set in Z. Since, g is an iα-continuous, then $g^{-1}(V)$ is an iα-open set in Y. Since, f is an iα-totally continuous, then $f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V)$ is clopen set in X. Therefore, $gof: X \to Z$ is totally continuous■

Definition 4.10 Let X, Y be a topological spaces, a mapping $f: X \to Y$ is said to be iα-contra-continuous (resp. i-contra-continuous), if the inverse image of every open subset of Y is an iα-closed (resp. i-closed) set in X.

Example 4.11 Let X=Y={a,b,c},τ={Ø,{a},X}, σ={Ø,{c},Y} and iα$O(X)$={Ø,{a},{b},{c},{a,b},{a,c},{b,c},X}. Clearly, the identity mapping $f: X \to Y$ is an i-contra-continuous and iα-contra-continuous.

Proposition 4.12 Every contra-continuous mapping is an i-contra-continuous.

Proof. Let $f: X \to Y$ be contra continuous mapping and V any open set in Y. Since, f is contra continuous, then $f^{-1}(V)$ is closed sets in X. Since, every closed set is an i-closed set, then $f^{-1}(V)$ is an i-closed set in X. Therefore, f is an i-contra-continuous■ Similarly we have the following results.

Proposition 4.13 Every contra semi-continuous mapping is an i-contra-continuous.

Proof. Clear since every semi-open set is an i-open set■

Proposition 4.14 Every contra α-continuous mapping is an i-contra-continuous.

Proof. Clear since every α-open set is an i-open set■ The converse of the propositions 4.12, 4.13 and 4.14 need not be true in general as shown in the following example

Example 4.15 Let X=Y={a,b,c}, τ={Ø,{a,c},X}, $iO(X)$={Ø,{a},{c},{a,b},{a,c},{b,c}, X} and σ={Ø,{c},Y}. Clearly, the identity mapping $f : X \to Y$ is is an i-contra continuous, but f is not contra-continuous, f is not contra semi-continuous, f is not contra α-continuous because for open subset f^{-1}{c}={c} is not closed in X, f^{-1}{c}={c}is not semi-closed in X and f^{-1}{c}={c}is not α-closed in X.

Proposition 4.16 Every i-contra-continuous mapping is an iα-contra-continuous.

Proof. Let $f : X \to Y$ be an i-contra continuous mapping and V any open set in Y. Since, f is an i-contra continuous, then $f^{-1}(V)$ is an i-closed sets in X. Since, every i-closed set is an iα-closed, then $f^{-1}(V)$ is an iα-closed set in X. Therefore, f is an iα-contra-continuous■

Remark 4.17 The following example shows that iα-contra-continuous mapping need not be contra-continuous, contra semi-continuous, contra-α-continuous and i-contra-continuous mappings.

Example 4.18 Let X=Y={a,b,c}, τ={Ø,{a},X}, SO(X)=$αO(X)$=$iO(X)$={Ø,{a},{a,b},{a,c},X}, iα$O(X)$= {Ø,{a},{b},{c},{a,b},{a,c},{b,c},X}, σ={Ø,{c}, Y}. A mapping $f : X \to Y$ is defined by f (a)=c, f(b)=b, f(c)=a. Clearly, f is an iα-contra-continuous, but f is not contra-continuous, f is not contra semi continuous, f is not contra α-continuous and f is not i-contra-continuous because for open subset {c}, f^{-1}{c}={a} is not closed,f^{-1}{c}={a} is not semi-closed , f^{-1}{c}={a} is not α-closed and f^{-1}{c}={a} is not i-closed in X.

Theorem 4.19 Every totally continuous mapping is an iα-contra continuous.

Proof. Let $f : X \to Y$ be totally continuous and V be any open set in Y. Since, f is totally continuous mapping, then $f^{-1}(V)$ is clopen set in X, and hence closed, it follows iα-closed set. Therefore, f is an iα-contra-continuous■ The converse of the above theorem need not be true as shown in the following example

Example 4.20 Let X=Y={a,b,c}, τ={Ø,{c},X}, σ={Ø,{a},Y} and iα$O(X)$={Ø,{a},{b},{c},{a,b},{a,c},{b,c},X}. Clearly, the identity mapping $f : X \to Y$ is an iα-contra-continuous, but f is not totally continuous because for open subset f^{-1}{a}={a} $\notin$ C$O(X)$.

5 Separation Axioms with iα-open Set

In this section, we introduce some new weak of separation axioms with iα-open sets.

Definition 5.1 A topological space X is said to be
(i) iα-T_o if for each pair distinct points of X, there exists iα-open set containing one point but not the other.
(ii) iα-T_1 (resp. clopen -T_1 [3]) if for each pair of distinct points of X, there exists two iα-open (resp. clopen) sets containing one point but not the other .
(iii)iα-T_2 (resp. ultra hausdorff (UT_2)[10]) if for each pair of distinct points of X can be separated by disjoint iα-open (resp. clopen) sets.

(iv) iα–regular (resp. ultra regular [9]) if for each closed set F not containing a point in X can be separated by disjoint iα-open (resp. clopen) sets.
(v) clopen regular [10] if for each clopen set F not containing a point in X can be separated by disjoint open sets.
(vi) iα–normal (resp. ultra normal[10], s-normal[13], α-normal[2]) if for each of non-empty disjoint closed sets in X can be separated by disjoint iα-open (resp. clopen, semi-open, α-open) sets.
(vii) clopen normal [10] if for each of non-empty disjoint clopen sets in X can be separated by disjoint open sets.
(viii) iα-$T_{1/2}$ if every iα-closed is i-closed in X.

Remark 5.2 The following example shows that iα-normal need not be normal, s-normal, α-normal spaces

Example 5.3 Let X={1,2,3,4,5}, τ={Ø,{1,2,3},{1,2,3,4},{1,2,3,5},X} and iαO(X)= {Ø,{1},{2},{3},{4},{5},{1,2},{1,3},{1,4},{1,5},{2,3},{2,4},{2,5},{3,4},{3,5},{4,5}, {1,2,3},{1,2,4},{1,2,5},{1,3,4},{1,3,5},{1,4,5},{2,3,4},{2,3,5},{2,4,5},{3,4,5},{1,2,3,4 },{1,2,3,5},{1,3,4,5},{1,2,4,5},{2,3,4,5},X}. Clearly, the space X is iα-T_o, iα-T_1, iα-T_2, iα–regular, iα–normal and iα-$T_{1/2}$, but X is not normal, s-normal and α-normal.

Theorem 5.4 if a mapping $f: X \to Y$ is an iα-contra–continuous mapping and the space X is an iα-$T_{1/2}$, then f is an i-contra-continuous.

Proof. Let $f: X \to Y$ iα-contra–continuous mapping and V is any open set in Y. Since, f is an iα-contra-continuous mapping, then $f^{-1}(V)$ is an iα-closed in X. Since, X is an iα-$T_{1/2}$, then $f^{-1}(V)$ is i-closed in X. Therefore, f is an i-contra-continuous■

Theorem 5.5 If $f: X \to Y$ is an iα-totally continuous injection mapping and Y is an iα-T_1, then X is clopen-T_1.

Proof. Let x and y be any two distinct points in X. Since, f is an injective, we have $f(x)$ and $f(y) \in Y$ such that $f(x) \neq f(y)$. Since, Y is an iα-T_1, there exists iα-open sets U and V in Y such that $f(x) \in U$, $f(y) \notin U$ and $f(y) \in V$, $f(x) \notin V$. Therefore, we have $x \in f^{-1}(U)$, $y \notin f^{-1}(U)$ and $y \in f^{-1}(V)$ and $x \notin f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are clopen subsets of X because f is an iα-totally continuous. This shows that X is clopen-T_1■

Theorem 5.6 If $f: X \to Y$ is an iα-totally continuous injection mapping and Y is an iα-T_o, then X is ultra-Hausdorff (UT_2).

Proof. Let a and b be any pair of distinct points of X and f be an injective, then $f(a) \neq f(b)$ in Y. Since Y is an iα-T_o, there exists iα-open set U containing $f(a)$ but not $f(b)$, then we have $a \in f^{-1}(U)$ and $b \notin f^{-1}(U)$. Since, f is an iα-totally continuous, then $f^{-1}(U)$ is clopen in X. Also $a \in f^{-1}(U)$ and $b \in X\text{-}f^{-1}(U)$. This implies every pair of distinct points of X can be separated by disjoint clopen sets in X. Therefore, X is ultra-Hausdorff■

Theorem 5.7 Let $f: X \to Y$ be a closed iα-continuous injection mapping. If Y is an iα–normal, then X is an iα–normal.

Proof. Let F_1 and F_2 be disjoint closed subsets of X. Since, f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since, Y is an iα–normal , $f(F_1)$ and $f(F_2)$ are separated by disjoint iα-open sets V_1 and V_2 respectively. Therefore, we obtain, $F_1 \subset f^{-1}(V_1)$ and $F_2 \subset f^{-1}(V_2)$. Since, f is an iα-continuous, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are iα-open sets in X. Also, $f^{-1}(V_1) \cap f^{-1}(V_1) = f^{-1}(V_1 \cap V_2) =$ Ø. Thus, for each pair of non-empty disjoint closed sets in X can be separated by disjoint iα-open sets. Therefore, X is an iα–normal ■

Theorem 5.8 If $f : X \rightarrow Y$ is an iα-totally continuous closed injection mapping and Y is an iα–normal, then X is ultra-normal.

Proof. Let F_1 and F_2 be disjoint closed subsets of X. Since, f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since, Y is an iα–normal , $f(F_1)$ and $f(F_2)$ are separated by disjoint iα-open sets V_1 and V_2 respectively. Therefore, we obtain, $F_1 \subset f^{-1}(V_1)$ and $F_2 \subset f^{-1}(V_2)$. Since, f is an iα-totally continuous, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are clopen sets in X. Also, $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = \emptyset$. Thus, for each pair of non-empty disjoint closed sets in X can be separated by disjoint clopen sets in X. Therefore, X is ultra-normal■

Theorem 5.9 Let $f : X \rightarrow Y$ be a totally continuous closed injection mapping, if Y is an iα–regular, then X is ultra-regular.

Proof. Let F be a closed set not containing x. Since, f is closed, we have $f(F)$ is a closed set in Y not containing $f(x)$. Since, Y is an iα–regular, there exists disjoint iα-open sets A and B such that $f(x) \in A$ and $f(F) \subset B$, which imply $x \in f^{-1}$(A) and $F \subset f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are clopen sets in X because f is totally continuous. Moreover, since f is an injective, we have $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$. Thus, for a pair of a point and a closed set not containing a point in X can be separated by disjoint clopen sets. Therefore, X is ultra-regular■

Theorem 5.10 If $f : X \rightarrow Y$ is totally continuous injective iα-open mapping from a clopen regular space X into a space Y, then Y is an iα–regular.

Proof. Let F be a closed set in Y and $y \notin F$. Take $y = f(x)$. Since, f is totally continuous, $f^{-1}(F)$ is clopen in X. Let $G = f^{-1}(F)$, then we have $x \notin G$. Since, X is clopen regular, there exists disjoint open sets U and V such that $G \subset U$ and $x \in V$. This implies $F = f(G) \subset f(U)$ and $y = f(x) \in V$. Further, since f is an injective and iα-open, we have $f(U) \cap f(V) = f(U \cap V) = f(\emptyset) = \emptyset$, $f(U)$ and $f(V)$ are an iα-open sets in Y. Thus, for each closed set F in Y and each y□F, there exists disjoint iα-open sets $f(U)$ and $f(V)$ in Y such that $F \subset f(U)$ and $y \in f(V)$. Therefore, Y is an iα–regular■

Theorem 5.11 If $f : X \rightarrow Y$ is a totally continuous injective and iα-open mapping from clopen normal space X into a space Y, then Y is an iα–normal.

Proof. Let F_1 and F_2 be any two disjoint closed sets in Y. Since, f is totally continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are clopen subsets of X. Take $U = f^{-1}(F_1)$ and $V = f^{-1}(F_2)$. Since, f is an injective, we have $U \cap V = f^{-1}(F_1) \cap f^{-1}(F_2) = f^{-1}(F_1 \cap F_2) = f^{-1}(\emptyset) = \emptyset$. Since, X is clopen normal, there exists disjoint open sets A and B such that $U \subset A$ and $V \subset B$. This implies $F_1 = f(U) \subset f(A)$ and $F_2 = f(V) \subset f(B)$. Further, since f is an injective iα-open, then $f(A)$ and $f(B)$ are disjoint iα-open sets. Thus, each pair of disjoint closed sets in Y can be separated by disjoint iα-open sets. Therefore, Y is an iα–normal■

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