Small Prime Modules and Small Prime Submodules

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Abstract

Let R be a commutative ring with identity, and M be a unital (left) R-module. In this paper we introduce and study the concepts: small prime submodules and small prime modules as generalizations of prime submodules and prime modules.

Among the results that we obtain is the following:
An R-module M is small prime if and only if the R-module R/annM is cogenerated by every non-trivial small submodule of M.

1. Introduction

An R-module M is called a prime module if \( \text{ann}_R M = \text{ann}_R N \) for each non-zero submodule N of M, [1]. A proper submodule N of M is called prime submodule if whenever \( r \in R \), \( x \in M \) and \( rx \in N \) implies either \( x \in N \) or \( r \in [N:M] \), [2]. As a generalization of these concepts, we introduce the concepts of small prime modules and small prime submodules.

Where we call M a small prime module if \( \text{ann} M = \text{ann} N \) for each non-zero small submodule N of M. And we call a submodule N of M small prime submodule if whenever \( r \in R \), \( x \in M \) with \( (x) \) is small in M and \( rx \in N \) implies either \( x \in N \) or \( r \in [N:M] \). Where a submodule N of M is called small (notationally, \( N \triangleleft M \)) if \( N + K = M \) for all submodules K of M implies \( K = M \), [3].

The main goal of this research is to study small prime modules and small prime submodules.

This research consists of two sections, in the first section we establish some properties of small prime submodules, and in the second section we give a comprehensive study of small prime modules.

2. Small Prime Submodules

We introduce and study the following concept:

2.1 Definition:

A proper submodule N of an R-module M is called small prime submodule if and only if whenever \( r \in R \) and \( x \in M \) with \( (x) \triangleleft M \) and \( rx \in N \) implies either \( x \in N \) or \( r \in [N:M] \).

A proper ideal I of a ring R is called small prime if I is a small prime submodule of the R-module R. Equivalent a proper ideal I of R is small prime if and only if whenever \( r, s \in R \) with \( r \triangleleft R \) and \( rs \in I \) implies either \( r \in I \) or \( s \in I \).

2.2 Examples and Remarks:

1. Every prime submodule is small prime and the converse is not true in general

For the converse, consider \( M = \mathbb{Z}_{24} \) as a \( \mathbb{Z} \)-module and \( N = (6) = \{0, 6, 12, 18\} \). N is a small prime submodule of M which is not prime.

To show that N is small prime in M, note that the small submodules of M are:
- \( (0) \), \( (6) \), and \( (12) \) \( \subseteq M \).

(i) \( 0 \in N, 0 \not\in (6) \)
(ii) \( 0 \not\in (12) \)
(iii) \( (6) \subseteq (12) \)

In the same way we test the elements \( 6, 12 \) and \( 18 \) of N. So N is small prime in M.

2. If M is an R-module in which every cyclic submodule is small, then every small prime submodule of M is prime.

3. If M is a hollow R-module, then a submodule N of M is small prime iff N is prime.

4. Every proper ideal of the ring \( \mathbb{Z} \) is small prime.

5. (0) is the only small prime submodule of the \( \mathbb{Z} \)-module \( \mathbb{Q} \), since for each \( x \in \mathbb{Q} \), \( (x) \triangleleft \mathbb{Q} \), so if N is a non-trivial submodule
of Q and rx∈N where r∈Z, then it is not necessarily that x∈N or r∈[N:Q].

6. If N is a small prime submodule of M, then it is not necessary that:
(i) \( \text{ann}_N(M) \) is a prime ideal of R.
(ii) \([N:M] \) is a prime ideal of R.
For example: Take \( M = Z_{24} \) as a Z-module and N= \( (\overline{6}) \). Then \( \text{ann}_Z(\overline{6}) = 4Z \) is not a prime ideal of Z and \([N:M] = 6Z \) is also not a prime ideal of Z.

7. A submodule of a small prime submodule need not be small prime.
For example: Let N = \( (\overline{6}) \). N is a small prime submodule of the Z-module \( Z_{24} \). Let \( K = (\overline{12}) = \{\overline{0}, \overline{12}\} \). K is not a small prime submodule of N, since \( (\overline{6}) \not\subseteq Z_{24} \) and \( \overline{12} = 2 \cdot \overline{6} \in K \) but \( \overline{6} \notin K \) and \( 2 \notin [K:Z_{24}] = 12Z \).

8. A direct summand of a small prime submodule is not in general small prime submodule.
For example: Let \( N = (\overline{2}) \) in the Z-module \( Z_{24} \). N is a small prime submodule of \( Z_{24} \) and \( N = (\overline{6}) \oplus (\overline{8}) \), while \( (\overline{8}) \) is not a small prime submodule of N, since \( (\overline{0}) \) and \( (\overline{12}) \) are the only small submodules of N with \([N:R] = 4Z \) and \( \overline{0} = 2 \cdot \overline{12} \in (\overline{8}) \) but \( 2 \notin [\overline{8}:N] \) and \( 12 \notin (\overline{8}) \).

9. If K ⊆ N are submodules of M and N is small prime in N, then it is not necessary that K is small prime in N.
For example: Let \( K = (\overline{12}) \) and \( N = (\overline{2}) \) in the Z-module \( Z_{24} \). N is small prime in \( Z_{24} \) but K is not small prime in N.

10. Let M be an R-module and I be an ideal of R such that \( I \subseteq \text{ann}M \). Let N be a submodule of M. Then N is small prime R-submodule of M iff N is a small prime R/I submodule of M.

Proof:
Let \( r \in R/I \) and \( x \in M \) with \( (x) \subseteq M \) and \( \overline{r}x \in N \). But \( \overline{r}x = rx \). Hence the result follows easily.

2.3 Proposition:
Let N and K be small prime submodules of an R-module M such that \([N:M] = [K:M] \). Then N∩K is a small prime submodule of M.

Proof:
Let \( r \in R \), \( x \in M \) with \( (x) \subseteq M \) and \( rx \in N \cap K \). Then \( rM \subseteq N \) and \( rx \in K \). Therefore \( (x \in N \) or \( r \in [N:M] \)) and \( (x \in K \) or \( r \in [K:M] \)). Hence \( (x \in N \) and \( x \in K \) or \( r \in [N:M] \) = [K:M], which implies \( x \in N \cap K \) or \( r \in [N \cap K:M] \).
Then N∩K is small prime submodule of M.

2.4 Proposition:
If N is a small prime submodule of an R-module M and I is any ideal of R, then \( [N : M/I] \) is a small prime submodule of M.

Proof:
Put \([N : M/I] = K \). Let \( r \in R \), \( x \in M \) with \( (x) \subseteq M \) and \( rx \in K \). Then \( rM \subseteq N \). That is \( rM \subseteq N \) \( \forall r \in R \). But \( (x) \subseteq M \) implies \( (x) \subseteq M \). \( ([4],[prop.1.1.3]) \). Therefore either \( x \in N \) \( \forall a \in I \) or \( r \in [N:M] \subseteq [K:M] \). So, either \( x \in N \) \( \forall a \in I \) or \( r \in [K:M] \). Thus either \( x \in K \) or \( r \in [K:M] \) which is what we wanted.

2.5 Remark:
The converse of proposition (2.4) is not true in general.
For example: Let \( M = Z_{24} \) as a Z-module, \( I = 2Z \) and \( N = \langle \overline{12} \rangle \). Then \([N : M] = \langle \overline{12} \rangle \cap Z_{24} = \langle \overline{6} \rangle \) is a small prime submodule of \( Z_{24} \), however N is not a small prime submodule of \( Z_{24} \).

2.6 Proposition:
Let \( f:M \longrightarrow M' \) be an R-epimorphism and let N be a small prime submodule of M'. Then \( f^{-1}(N) \) is a small prime submodule of M.

Proof:
Let \( r \in R \), \( x \in M \), \( (x) \subseteq M \) and \( rx \in f^{-1}(N) \). Then \( f(rx) = rf(x) \in N \). But \( (x) \subseteq M \) implies \( (f(x)) \subseteq M' \). \([4],[prop.1.1.3] \). And N is small prime in M' implies either \( f(x) \in N \) or \( r \in [N:M'] \). If \( f(x) \in N \), then \( x \in f^{-1}(N) \) and \( r \in f^{-1}(N) \). If \( r \in [N:M'] \), then \( rM' \subseteq N \) and hence \( rf(M) = f(rM) \subseteq N \) which gives \( rM \subseteq f^{-1}(N) \), that is \( r \in [f^{-1}(N):M] \). Hence \( f^{-1}(N) \) is a small prime submodule of M.
2.7 Remark:

A homomorphic image of a small prime submodule is not in general small prime. For example: let \( f: \mathbb{Z}_{24} \rightarrow \mathbb{Z}_{24} \) be such that \( f(\bar{x}) = 2\bar{x} \ \forall \ \bar{x} \in \mathbb{Z}_{24} \), The submodule \( \langle 2 \rangle \) is small prime in \( \mathbb{Z}_{24} \), while \( f(\langle 2 \rangle) = \langle 4 \rangle \) is not a small prime submodule of \( \mathbb{Z}_{24} \).

2.8 Proposition:

Let \( M \) be a f.g. faithful multiplication \( R \)-module. If \( N \) is a small prime submodule of \( M \), then \( [N:M] \) is a small prime ideal of \( R \).

**Proof:**

Let \( r, a \in R \) with \( (a) \subseteq R \) and \( ra \in [N:M] \). Then \( (ra)M \subseteq N \). But \( (ra) \subseteq (a) \subseteq R \) implies that \( ra \subseteq R \), \((4), prop. 1.1.3\) and since \( M \) is f.g. faithful multiplication \( R \)-module, therefore \( ra \subseteq M \). Hence either \( r \in [N:M] \) or \( a \in [N:M] \) and hence \( [N:M] \) is a small prime ideal of \( R \).

3. Small Prime Modules

In this section, we give and study a generalization of a prime module which is a small prime module. Many properties and characterizations of this concept are obtained. Moreover, we study the relationships between small prime modules and special kinds of modules.

3.1 Definition:

An \( R \)-module \( M \) is called small prime if and only if \( \text{ann}M = \text{ann}N \) for each non-zero small submodule \( N \) of \( M \).

A ring \( R \) is called small prime iff \( R \) is a small prime \( R \)-module. Equivalently, \( R \) is small prime iff \( \text{ann}_R I = 0 \) for each non-zero small ideal \( I \) of \( R \).

3.2 Examples and Remarks:

1. Every prime \( R \)-module is small prime but not conversely in general.

   For the converse, \( Z_6 \) as a \( Z \)-module is small prime but not prime.

2. \( Z_4 \) as a \( Z \)-module is not small prime, since \( \langle 2 \rangle \not\subseteq Z_4 \) and \( \text{ann}_Z(Z_4) = 2Z \neq \text{ann}_Z(Z_4) \).

3. Every semisimple \( R \)-module is small prime but not conversely, since \( Z \) as a \( Z \)-module is small prime but not semisimple.

4. \( Z_{p^n} \) and \( \mathbb{Q}/\mathbb{Z} \) as \( Z \)-module are not small prime.

5. Every hollow small prime \( R \)-module is prime.

6. Every integral domain is a small prime ring.

7. Let \( M \) be an \( R \)-module and \( I \) be an ideal of \( R \) such that \( I \subset \text{ann}M \). Then \( M \) is a small prime \( R \)-module iff \( M \) is a small prime \( R/I \)-module.

8. Let \( M_1 \) and \( M_2 \) be isomorphic \( R \)-modules. Then \( M_1 \) is small prime iff \( M_2 \) is small prime.

**Proof (8):**

Assume that \( M_1 \) is small prime and let \( \varphi: M_1 \rightarrow M_2 \) be an \( R \)-isomorphism. Let \( 0 \neq N \subseteq M_2 \). We have to show that \( \text{ann}M_2 = \text{ann}N \). \( \varphi^{-1}(N) \subseteq M_1 \) and \( \varphi^{-1}(N) \neq 0 \). So, \( \text{ann}M_1 = \text{ann}\varphi^{-1}(N) \). But \( M_1 \cap M_2 \) implies that \( \text{ann}M_1 = \text{ann}M_2 \), \([5]\). And we can show easily that \( \text{ann}N = \text{ann}\varphi^{-1}(N) \), which completes the proof.

Now, we give some characterizations of small prime modules.

3.3 Proposition:

An \( R \)-module \( M \) is small prime iff \( \text{ann}M = \text{ann}(m) \ \forall 0 \neq m \in M \) such that \( (m) \subseteq M \).

**Proof:**

\( \Rightarrow \) Is obvious.

\( \Leftarrow \) Let \( 0 \neq N \subseteq M \) and let \( r \in \text{ann}N \). Then \( rm = 0 \ \forall m \in N \) and hence \( r \in \text{ann}(m) \ \forall m \in N \). But \( (m) \subseteq N \subseteq M \) implies that \( (m) \subseteq M \), \([4]\). So, \( \text{ann}M = \text{ann}(m) \) (by hyp.). But \( \text{ann}N \subseteq \text{ann}(m) \) therefore \( \text{ann}N \subseteq \text{ann}M \). Thus \( \text{ann}M = \text{ann}N \) and hence \( M \) is small prime.

3.4 Proposition:

An \( R \)-module \( M \) is small prime iff \( (0) \) is a small prime submodule of \( M \).

**Proof:**

\( \Rightarrow \) To prove \( (0) \) is a small prime submodule of \( M \). Let \( r \in R \), \( x \in M \) with \( (x) \subseteq M \) and \( rx = 0 \). If \( x \neq 0 \), then \( \text{ann}(x) = \text{ann}M \) (since \( M \) is small prime), and hence \( r \in \text{ann}M = [(0):M] \).

If \( x = 0 \), then \( x \in (0) \). Thus \( (0) \) is a small prime submodule of \( M \).

\( \Leftarrow \) If \( (0) \) is a small prime submodule of \( M \). Let \( 0 \neq N \subseteq M \) and let \( r \in \text{ann}N \). Then \( rx = 0 \ \forall x \in N \). Therefore \( rx \in (0) \). Assume that \( x \neq 0 \), then \( r \in [(0):M] = \text{ann}M \). Hence \( \text{ann}N \subseteq \text{ann}M \).
therefore \( \text{ann} M = \text{ann} N \). That is \( M \) is small prime.

### 3.5 Corollary:

A proper submodule \( N \) of an \( R \)-module \( M \) is small prime submodule iff \( M/N \) is a small prime \( R \)-module.

### 3.6 Corollary:

Let \( M \) be an \( R \)-module. Then the following statement are equivalent:

1. \( M \) is small prime.
2. \( \text{ann} M = \text{ann}(m) \) \( \forall 0 \neq m \in M \) and \( (m) \subseteq M \).
3. \( 0 \) is a small prime submodule of \( M \).

The following is another characterization of small prime module.

### 3.7 Proposition:

An \( R \)-module \( M \) is small prime iff for each non-zero element \( m \in M \) with \( (m) \subseteq M \), there exists an \( R \)-isomorphism \( f:(m) \rightarrow \bar{R} = R/\text{ann} M \) such that \( f(m) = \bar{1} \).

**Proof:**

\( \Rightarrow \) Assume that \( M \) is a small prime \( R \)-module. Let \( 0 \neq m \in M \) and \( (m) \subseteq M \). Define \( f:(m) \rightarrow \bar{R} \) such that \( f(rm) = r \cdot \bar{1} \forall r \in R \). \( f \) is well defined, for if \( r_1m = r_2m; r_1, r_2 \in R \), and \( r_1 = r_2 \cdot \text{ann}(m) \). But \( M \) is small prime, therefore \( r_1 = r_2 \cdot \text{ann} M \) (by prop.3.3). Hence \( \bar{r}_1 = \bar{r}_2 \) implies \( \bar{r}_1 \cdot \bar{1} = \bar{r}_2 \cdot \bar{1} \). It can be easily checked that \( f \) is an \( R \)-isomorphism. Moreover, for each \( r \in R \), \( f(rm) = r \cdot \bar{1} = r(1 + \text{ann} M) = r + \text{ann} M = \bar{r} \) and hence \( f(m) = f(1m) = \bar{r} \).

\( \Leftarrow \) To prove \( M \) is small prime

Let \( 0 \neq m \in M \) and \( (m) \subseteq M \). By hypothesis there exists an \( R \)-isomorphism \( f:(m) \rightarrow \bar{R} \) such that \( f(m) = \bar{1} \). Let \( r \in \text{ann}(m) \). Then \( rm = 0 \) and hence \( f(rm) = \bar{0} \).

Now, \( \bar{0} = f(rm) = rf(m) = r \cdot \bar{1} = \bar{r} \). Thus \( \bar{r} = \bar{0} \) implies that \( r \in \text{ann} M \). Therefore \( \text{ann}(m) \subseteq \text{ann} M \) and hence \( \text{ann} M = \text{ann} (m) \). Thus \( M \) is small prime (by prop.3.3).

The following are some consequences of prop.(3.7).

### 3.8 Corollary:

An \( R \)-module \( M \) is small prime iff every non-trivial cyclic small submodule of \( M \) is isomorphic to the \( R/\text{ann} M \).

### 3.9 Corollary:

An \( R \)-module \( M \) is small prime iff all non-trivial cyclic small submodules of \( M \) are isomorphic to each other.

### 3.10 Corollary:

A faithful \( R \)-module \( M \) is small prime iff every non-trivial cyclic small submodule of \( M \) is isomorphic to the \( R \)-module \( R \).

It is well-known that: if \( M \) is a prime \( R \)-module, then \( \text{ann} M \) is a prime ideal of \( R \). This fact does not hold for small prime modules, for instance, the \( Z \)-module \( Z_6 \) small prime, but \( \text{ann} Z_6 = 2Z \) is not a prime ideal of \( Z \). However we have the following restricted result:

### 3.11 Proposition:

If \( M \) is a small prime \( R \)-module, then \( \text{ann} N \) is a prime ideal of \( R \) for each non-trivial small submodule \( N \) of \( M \).

**Proof:**

Let \( 0 \neq N \subseteq M \). Let \( a, b \in R \) and \( ab \in \text{ann} N \). Then \( abN = 0 \). Suppose that \( bN \neq 0 \). But \( bN \) is a submodule of \( N \) and \( N \subseteq M \) implies that \( bN \subseteq M \), [4]. Therefore \( \text{ann} M = \text{ann} N \) and since \( a \in \text{ann} N \), then \( a \in \text{ann} M \). On the other hand \( \text{ann} M = \text{ann} N \), so \( a \in \text{ann} N \). Therefore \( \text{ann} N \) is a prime ideal of \( R \).

Note that the converse of prop.(3.11) is not true in general.

For example, \( Z_4 \) as a \( Z \)-module is not small prime while \( (2) \subseteq Z_4 \) and \( \text{ann} (2) = 2Z \) is a prime ideal of \( Z \).

### 3.12 Proposition:

A non-trivial submodule of a small prime \( R \)-module is also a small prime \( R \)-module.

**Proof:**

Let \( M \) be a small prime \( R \)-module and \( N \neq 0 \) be a submodule of \( M \). Let \( 0 \neq K \subseteq N \). Then \( K \subseteq M \), [4]. Therefore \( \text{ann} M = \text{ann} K \). But \( \text{ann} M \subseteq \text{ann} N \). Therefore \( \text{ann} K \subseteq \text{ann} N \). So, \( \text{ann} N = \text{ann} K \) and hence \( N \) is a small prime \( R \)-module.

### 3.13 Corollary:

If \( M \) is an \( R \)-module whose injective hull \( E(M) \) is small prime, then \( M \) is also small prime.

Note that the converse of coro. (3.13) is not true in general, for example \( Z_2 \) as a \( Z \)-module is small prime and \( E(Z_2) = Z_2 \) is not small prime \( Z \)-module.
3.14 Corollary:
A non-trivial direct summand of a small prime R-module is a small prime R-module.

The converse of prop.(3.14) need not be true in general.

Consider the following example:
Let \( M = \mathbb{Z} \oplus \mathbb{Z}_4 \) as a \( \mathbb{Z} \)-module. \( M \) is not small prime, while \( N = \mathbb{Z} \oplus 0 \) is a submodule of \( M \) and \( N \) is a small prime \( \mathbb{Z} \)-module.

However the following result holds:

3.15 Proposition:
Let \( M \) be an R-module such that \( \text{Rad} M \) is a proper direct summand of \( M \). If \( \text{Rad} M \) is a small prime R-module and \( \text{ann} M = \text{ann} \text{Rad} M \), then \( M \) is a small prime R-module.

Proof:
Let \( 0 \neq m \in M \) and \( (m) \not\subseteq M \). Then \( m \in \text{Rad} M \), [4]. Hence \( (m) \not\subseteq \text{Rad} M \). Therefore \( \text{ann} \text{Rad} M = \text{ann}(m) \) and hence \( \text{ann} M = \text{ann}(m) \). So \( M \) is small prime (by prop.3.3).

3.16 Proposition:
Let \( M = M_1 \oplus M_2 \) be an R-module such that \( \text{ann} M_1 + \text{ann} M_2 = R \). Then \( M \) is small prime R-module iff \( M_1 \) and \( M_2 \) are small prime R-modules.

Proof:
\[ \Rightarrow \] To prove \( M \) is small prime.
Let \( 0 \neq N \subseteq M \). Since \( \text{ann} M_1 + \text{ann} M_2 = R \), then \( N = N_i \oplus N_2 \) where \( N_i \) is a submodule of \( M_i \) respectively for \( i = 1,2,6 \). But \( N \not\subseteq M \), therefore \( N_i \not\subseteq M_i \) respectively for \( i = 1,2,6 \). Now, \( \text{ann} N = \text{ann}(N_1 \oplus N_2) = \text{ann} N_1 \cap \text{ann} N_2 = \text{ann} M_1 \cap \text{ann} M_2 \) (since \( M_1 \) and \( M_2 \) are small prime). Therefore \( \text{ann} N = \text{ann}(M_1 \oplus M_2) = \text{ann} M \). Hence \( M \) is small prime.
\[ \Leftarrow \] Follows by coro.(3.14).

3.17 Proposition:
Let \( M \) and \( M' \) be two R-modules such that \( \text{ann} M = \text{ann} M' \). If \( f:M\rightarrow M' \) is an R-homomorphism and \( M' \) is small prime then \( M \) is also small prime.

Proof:
Let \( 0 \neq N \subseteq M \) and \( r \in \text{ann} N \). Then \( rN = 0 \). \( f(rN) = r(f(N)) = 0 \) implies \( r \in \text{ann} f(N) \). But \( N \subseteq M \) gives \( f(N) \subseteq M' \), [4]. Therefore \( r \in \text{ann} M' \) (since \( M' \) is small prime). But \( \text{ann} M = \text{ann} M' \), so \( r \in \text{ann} M \) and hence \( \text{ann} N \subseteq \text{ann} M \). Therefore \( \text{ann} M = \text{ann} N \) which completes the proof.

3.18 Remark:
The condition \( \text{ann} M = \text{ann} M' \) in prop. (3.17) can not be dropped. Consider the following example:
Let \( M = Z_{12} \) and \( M' = Z_5 \) as \( \mathbb{Z} \)-modules. Define \( f:Z_{12}\rightarrow Z_5 \) such that \( f(x) = 5x \forall x \in Z_{12} \).
Clearly, \( f \) is a \( \mathbb{Z} \)-homomorphism. \( Z_5 \) is a small prime \( \mathbb{Z} \)-module, while \( Z_{12} \) is not a small prime \( \mathbb{Z} \)-module, since \( (6) \not\subseteq Z_{12} \) but \( \text{ann} Z_{12} = 12 \mathbb{Z} \neq 2 \mathbb{Z} = \text{ann}(6) \).

3.19 Remark:
Epimorphic image of a small prime R-module need not be small prime in general. Consider the following example:
Let \( \mathbb{Z} \) and \( \mathbb{Z}_4 \) be \( \mathbb{Z} \)-modules and \( \pi: \mathbb{Z} \rightarrow \mathbb{Z}_4 \) be the natural homomorphism. \( \mathbb{Z} \) is a prime \( \mathbb{Z} \)-module and hence small prime, while \( \mathbb{Z}_4 \) is not small prime \( \mathbb{Z} \)-module.

The following are some consequences of prop.(3.17):

3.20 Corollary:
Let \( N \) be a submodule of an R-module \( M \) such that \( \text{ann} M = [N:M] \). If \( M/N \) is small prime, then \( M \) is also small prime.

3.21 Corollary:
Let \( N \) be a small prime submodule of an R-module \( M \) such that \( \text{ann} M = [N:M] \). Then \( M \) is a small prime R-module.

Recall that an R-module \( M \) is called coprime if \( \text{ann} M = \text{ann}(M/N) \) for every proper submodule \( N \) of \( M \), [7]. \( \mathbb{Z} \) as a \( \mathbb{Z} \)-module is small prime but not coprime. \( Z_{p^r} \) as a \( \mathbb{Z} \)-module is coprime but not small prime.

3.22 Corollary:
Let \( M \) be a coprime R-module and \( N \) be a submodule of \( M \). If \( M/N \) is small prime R-module, then \( M \) is a small prime R-module.

3.23 Corollary:
Let \( M \) be a coprime R-module and \( N \) be a small prime submodule of \( M \). Then \( M \) is a small prime R-module.
3.24 Proposition:
Let \( N \) be a submodule of an \( R \)-module \( M \). If \( M/N \) is small prime, then \([N:M] = [N:K]\) for all small submodules \( K \) of \( M \) containing \( N \).

Proof:
Let \( 0 \neq K \triangleleft M \) and \( K \supseteq N \). Then \( K/N \triangleleft M/N \), ([4], prop.1.1.2). But \( M/N \) is small prime, so \( \text{ann}M/N = \text{ann}K/N \). Hence \([N:M] = [N:K]\).

3.25 Corollary:
If \( N \) is a small prime submodule of an \( R \)-module \( M \), then \([N:M] = [N:K]\) for all small submodules \( K \) of \( M \) containing \( N \).

3.26 Proposition:
If \( N \) is a small submodule of an \( R \)-module \( M \) such that \([N:M] = [N:K]\) for all small submodules \( K \) of \( M \) containing \( N \), then \( M/N \) is small prime.

Proof:
Let \( K \neq N \) be a submodule of \( M \) containing \( N \) such that \( K/N \triangleleft M/N \). Then \( K \triangleleft M \) ([4], prop.1.1.2). Hence \([N:M] = [N:K]\) and therefore \( \text{ann}M/N = \text{ann}K/N \). So \( M/N \) is small prime.

3.27 Corollary:
Let \( N \) be a small submodule of an \( R \)-module \( M \). Then \( M/N \) is small prime iff \([N:M] = [N:K]\) for all small submodules \( K \) of \( M \) containing \( N \).

3.28 Corollary:
Let \( N \) be a small submodule of an \( R \)-module \( M \). Then \( N \) is a small prime submodule of \( M \) iff \([N:M] = [N:K]\) for all small submodules \( K \) of \( M \) containing \( N \).

3.29 Corollary:
Let \( M \) be a hollow \( R \)-module and \( N \) be a submodule of \( M \). Then \( M/N \) is small prime iff \([N:M] = [N:K]\) for all submodules \( K \) of \( M \) containing \( N \).

3.30 Corollary:
Let \( M \) be a hollow \( R \)-module and \( N \) be a submodule of \( M \). Then \( N \) is a small prime submodule of \( M \) iff \([N:M] = [N:K]\) for all submodules \( K \) of \( M \) containing \( N \).

It is well-known that: If \( M \) is an \( R \)-module such that \( R/\text{ann}M \) is an integral domain and \( M \) is a torsion-free \( R/\text{ann}M \)-module, then \( M \) is a prime \( R \)-module, so under these conditions \( M \) is a small prime \( R \)-module, while the converse is not true in general, for instance \( Z_6 \) is a small prime \( Z \)-module and \( Z/6Z \) is not an integral domain, moreover \( Z_6 \) is not a torsion-free \( Z \)-module. However by using an extra condition we can prove that the converse holds.

3.31 Proposition:
Let \( M \) be a small prime \( R \)-module in which every non-trivial cyclic submodule is small. Then \( R/\text{ann}M \) is an integral domain and \( M \) is a torsion-free \( R/\text{ann}M \)-module.

Proof:
Let \( \bar{r}, \bar{s} \in R/\text{ann}M \). Suppose that \( \bar{r} \bar{s} = \bar{0} \). Then \( rs \in \text{ann}M \) and hence \( rs \in \text{ann}(m) \) for each \( 0 \neq m \in M \) (by prop.3.3). So, \( rs = 0 \) implies either \( r \in \text{ann}(sm) \) or \( s \in \text{ann}(rm) \). On the other hand \( \text{ann}(sm) = \text{ann}M = \text{ann}(rm) \). Therefore either \( r \in \text{ann}M \) or \( s \in \text{ann}M \), equivalently, either \( \bar{r} = \bar{0} \) or \( \bar{s} = \bar{0} \) and hence \( R/\text{ann}M \) is an integral domain.

Now, suppose that \( \bar{r}m = 0 \) with \( 0 \neq m \in M \) and \( \bar{r} \in R/\text{ann}M \). Then \( rm = 0 \) and hence \( r \in \text{ann}(m) = \text{ann}M \) (by prop.3.3). So, \( \bar{r} = \bar{0} \), whence \( M \) is a torsion-free \( R/\text{ann}M \)-module.

3.32 Corollary:
Let \( M \) be an \( R \)-module in which every non-trivial cyclic submodule is small. Then \( M \) is small prime iff \( R/\text{ann}M \) is an integral domain and \( M \) is a torsion-free \( R/\text{ann}M \)-module.

3.33 Corollary:
Let \( M \) be a faithful \( R \)-module in which every non-trivial cyclic submodule is small. Then \( M \) is small prime iff \( R \) is an integral domain and \( M \) is a torsion-free \( R/\text{ann}M \)-module.

3.34 Corollary:
Let \( M \) be a hollow \( R \)-module. Then \( M \) is small prime \( R \)-module iff \( R/\text{ann}M \) is an integral domain and \( M \) is a torsion-free \( R/\text{ann}M \)-module.

The following result is another characterization of small prime \( R \)-module \( M \) in terms of the \( R \)-module \( R/\text{ann}M \).

3.35 Theorem:
An \( R \)-module \( M \) is small prime iff the \( R \)-module \( R/\text{ann}M \) is a generated by every non-trivial small submodule of \( M \).
Proof: 
⇒ Suppose that M is small prime. Let 0 ≠ N ⊆ M and let 0 ≠ x ∈ N. Then 0 ≠ (x) ⊆ M, [4]. Therefore annM = ann(x) (by prop.3.3).

Now, R/annM = R/ann(x) ⊆ Rx = (x) is a submodule of N. Hence there exists a monomorphism from R/annM into N, whence R/annM is cogenerated by N, which proves the “only if” part.

⇐ To prove the “if” part.

Let 0 ≠ N ⊆ M. Then R/annM is cogenerated by N. So, there exists a monomorphism say, f: R/annM → N for some index set I. Let r ∈ annN. Then rN = 0.

Now, f((rN)) ∈ N and hence (f(r))(i) ∈ Ni ∩ N ∀i ∈ I. But f((r)) = r(f((r)))(i) = (rf((r)))(i) ∈ N. Hence (f((r)))(i) = 0 ∀i ∈ I, which implies that f((r)) = 0. But f is a monomorphism, therefore r = 0. Hence r ∈ annM which is what we wanted.

3.36 Corollary:
A faithful R-module M is small prime iff the R-module R is cogenerated by every non-trivial small submodule of M.

Now, we give the following lemma in order to discuss the localization of small prime module.

3.37 Lemma:
Let M be an R-module, S be a multiplicatively closed subset of R such that N ∩ M = 0 for each proper submodule N of M. Then N ∩ M if N ∩ M.

Proof: 
⇒ Let N ⊆ M.

Suppose that there exists a proper submodule K of M such that N ∩ K = M. Then there exists a proper submodule W of M such that K = W. So, N ∩ W = M, implies that N ∩ W = M (8),Ex.9.11(iii),p.173). Hence N + W = M. But N ∩ M, therefore W = M and W = M which is a contradiction.

⇐ Let N ∩ M.

Suppose that there exists a proper submodule W of M such that N + W = M. Therefore (N + W)S = M. Then N ∩ W = M implies that W = M which is a contradiction since W ≠ M.

3.38 Proposition:
Let M be a f.g. R-module. Then M is a small prime R-module if and only if M_P is a small prime R_P-module for each maximal (prime) ideal P of R.

Proof: 
⇒ Suppose that M is a small prime R-module. Let P be a maximal ideal of R and let 0 ≠ m/s ∈ M_P with m ∈ M and s ∈ P. Suppose that m/s ∈ M_P.

Then by lemma 3.37, we get that m/s ∉ M_P and hence ann(m) = annM. But M is f.g. therefore (ann(m))_P = (annM)_P, ([9], prop.3.14,p.43). So, ann(m/s) = ann(M_P) and M_P is a small prime R_P-module.

⇐ Follows similarly.

Recall that, an R-module M is called multiplication if for each submodule N of M there is an ideal I of R such that N = IM, [10].

Next, we study the relationships between small prime modules and multiplication modules.

3.39 Theorem:
Let M be a f.g. faithful multiplication R-module. Then M is a small prime R-module if R is a small prime ring.

Proof: 
⇒ Let I be a non-trivial small ideal of R. Then it can be shown easily that IM is a small submodule of M.

If IM = 0, then I ⊆ annM = 0 which is a contradiction.

Hence 0 ≠ IM ⊆ M and since M is small prime, then o = annM = annM. But annI ⊆ annM, therefore annI = 0 and hence R is a small prime ring.

⇐ Let 0 ≠ N ⊆ M. Then [N:M] ⊆ R, ([4],prop.1.1.8). But M being multiplication, implies that N = [N:M]M, [10]. Hence [N:M] ≠ 0 and since R is a small prime ring, then ann[N:M] = 0.

On the other hand M is faithful yields ann[N:M]M = ann[N:M] and hence annN= 0 which completes the proof.
3.40 Corollary:
Let $M$ be a cyclic faithful $R$-module. Then $M$ is small prime $R$-module iff $R$ is a small prime ring.

It is well known that: If $M$ is a multiplication $R$-module and $N$ is a submodule of $M$, then $N$ is an $R$-submodule of $M$ iff $N$ is an $S$-submodule of $M$ (where $S=\text{End}_R(M)$). Hence one can show easily that $N$ is a small $R$-submodule of $M$ iff $N$ is a small $S$-submodule of $M$.

Using this fact we can give the following results:

3.41 Proposition:
Let $M$ be a multiplication $R$-module. If $M$ is a small prime $S$-module, then $M$ is a small prime $R$-module (where $S=\text{End}_R(M)$).

Proof:
Let $0\neq N$ be a small $R$-submodule of $M$. Suppose that there exists $r\in\text{ann}_R N$ and $r \notin \text{ann}_SM$. Thus $rM \neq 0$. Define $f:M \rightarrow M$ by $f(m) = rm \forall m \in M$.

Clearly, $f$ is a well-defined $R$-homomorphism and $f \neq 0$. But $f(N) = rN = 0$ implies $f \in \text{ann}_S N = \text{ann}_SM$ (since $M$ is a small prime $S$-module). Hence $f(M) = 0$, that is $f = 0$ which is a contradiction. Thus $\text{ann}_N \subseteq \text{ann}_M$ and hence $M$ is a small prime $R$-module.

Recall that an $R$-module $M$ is called a scalar module if for all $f \in \text{End}_R(M); f \neq 0$ there exists $r \in R, r \neq 0$ such that $f(m) = rm \forall m \in M$, [11].

3.42 Proposition:
Let $M$ be a f.g. multiplication $R$-module. If $M$ is a small prime $R$-module, then $M$ is a small prime $S$-module.

Proof:
$M$ being a f.g. multiplication $R$-module implies that $M$ is a scalar $R$-module, ([11], prop.1.1.10). Let $N$ be a non-trivial small $S$-submodule of $M$. Then $N$ is a non-trivial small $R$-submodule of $M$. Suppose that there exists $f \in S, f \in \text{ann}_SN$ and $f \notin \text{ann}_SM$ such that $f(x) = rx \forall x \in M$. Thus $rN = 0$ and hence $r \in \text{ann}_NM = \text{ann}_SM$, so, $rM = 0$. Hence $f(M) = 0$ which is a contradiction. Therefore $\text{ann}_SM = \text{ann}_SN$ which completes the proof.

3.43 Corollary:
Let $M$ be a f.g. multiplication $R$-module. Then $M$ is a small prime $R$-module iff $M$ is a small prime $S$-module.

3.44 Proposition:
Let $M$ be a scalar $R$-module such that $\text{ann}M$ is a prime ideal of $R$. Then $S=\text{End}_R(M)$ is a small prime ring.

Proof:
$\text{ann}M$ being a prime ideal of $R$ implies that $R/\text{ann}M$ is an integral domain. But $M$ is a scalar $R$-module, implies that $S \cong R/\text{ann}M$ ([12],lemma 6.1,p.79). Thus $S$ is an integral domain. Hence $S$ is a small prime ring (by Ex. and Rem.3.2(6)).

3.45 Proposition:
Let $M$ be a faithful scalar $R$-module. Then $S = \text{End}_R(M)$ is a small prime ring iff $R$ is a small prime ring.

Proof:
$M$ is a scalar $R$-module implies that $S \cong R/\text{ann}M$ ([12],lemma 6.1,p.79). But $M$ being faithful $R$-module so, $S \cong R$. Therefore $R$ is a small prime iff $S$ is small prime (Ex. and Rem.3.2(8)).

3.46 Proposition:
Let $M$ be a faithful multiplication $R$-module. Then the following statements are equivalent:

1. $M$ is a small prime $R$-module.
2. $R$ is a small prime ring.
3. $S = \text{End}_R(M)$ is a small prime ring.

Proof:
(1) $\Leftrightarrow$ (2) by Th. 3.39.
(2) $\Leftrightarrow$ (3) since $M$ is f.g. multiplication $R$-module then $M$ is a scalar $R$-module ([11], Coro.1.1.11), so the result follows according to Prop.3.44.

References


الخلاصة

لنكن حلقة أدلية ذات محايد وليكن M مقاساً أحادياً (أي) على الحلقة R. قدمنا ودرسنا في هذا البحث المفاهيم: المقاسات الأولية الجزئية الصغيرة والمقاسات الأولية الصغيرة كأعمال للمقاسات الجزئية الأولية والمقياسات الأولية.

من بين النتائج التي حصلنا عليها:

- القياس R يكون أولي صغيرة إذا وفقط إذا كان القياس M على الحلقة R هو مقاس مضاد لكل مقاس جزئي صغيرة غير تأهَّل من M.